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ON THE EXISTENCE OF AN ARROW AND A BERGSON-SAMUELSON SOCIAL WELFARE FUNCTION

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Let Σ be the set of all possible preferences over a given set of alternatives A. Let Ω be a proper subset of Σ and let $P \in \Omega^n$ be a fixed profile of preferences. P *is heterogeneous* in Ω if for all $a, b, c \in A$ and $Q \in \Omega^n$, there exist three alternatives $x, y, z \in A$ such that $Q(a,b,c) = P(x,y,z)$ where $Q(B)$ denotes the subprofile over a set of alternatives $B \subset A$. An Arrow SWF f is a function $f: \Omega^n \to \Sigma$ satisfying the conditions Pareto and IIA. A Bergson-Samuelson SWF is a function $f: P \to \Sigma$ satisfying Pareto and Independence + Neutrality. The paper shows that (a) there exists a neutral nondictatorial Arrow SWF on Ω if and only if there exists a neutral nondictatorial Bergson-Samuelson SWF on P. (b) There exists a nondictatorial n person Bergson-Samuelson SWF on P if and only if there exists a 3 person Bergson-Samuelson SWF on P. (c) There exists a nondictatorial Arrow SWF on Ω if and only if there exists a nondictatorial Bergson-Samuelson SWF on P.

Key words: Bergson-Samuelson social welfare function; heterogeneous profile; restricted domain.

1. Introduction

The distinction between an Arrow and a Bergson-Samuelson social welfare function (SWF) is that the latter is defined on a single, fixed set of voter's preferences (profile of preferences) while the former is defined on any possible preference profiles which the voters might have.

Theorems about the nonexistence of both types of SWF's are well known. See Arrow (1963), Parks (1976), Kemp and Ng (1976), and Pollak (1979). All the results about the single profile case require the single profile to be extremely heterogenous.[1] Specifically, the profile P had to satisfy the requirement that for any logically possible subprofile $Q(a,b,c)$ over the alternatives a, b, c there exist alternatives x, y, z such that $P(x,y,z) = Q(a,b,c)$. Thus all the heterogeneity of the domain of preferences is imbedded in P.

The question that arises is what are the minimal restrictions on P which will insure an impossibility result or equivalently the weakest restrictions on the fixed profile P

[1] The heterogeneity condition is discussed in detail in Blair (1981).

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which will insure the existence of a Bergson-Samuelson SWF on P.

The purpose of this paper is to answer the above question with and without the neutrality condition. The first theorem shows that given a fixed profile P and a restricted domain Ω in which P is heterogenous, one can find a neutral nondictatorial Arrow SWF on the domain Ω if and only if one can find a neutral nondictatorial Bergson-Samuelson SWF on P.

The problem with relaxing the neutrality condition is that it is necessary for imposing the independence condition. If one was assured of the existence of a Bergson-Samuelson SWF for any number of voters, then an independence condition can be found which does not imply neutrality. The condition states that if two coalitions of equal size coincide with their vote on a pair, then the outcome of the Bergson-Samuelson SWF on these two coalition should be the same for this pair.

Theorem 3.2 assures us that in the neutral case, the existence of a neutral nondictatorial Bergson-Samuelson SWF is independent of the number of voters. The weaker independence condition is thus imposed in the nonneutral context. With a necessary change in the heterogeneity condition, Theorem 4.1 then reaffirms the result of Theorem 2.1 without the neutrality condition.

2. The existence issue

Pollak (1979) showed that if preferences are heterogenous enough in a given, fixed, profile, then there does not exist a Bergson-Samuelson SWF on this profile. In this section I ask the question of what is the minimal heterogeneity that will insure us of his result, or equivalently, what are the necessary and sufficient conditions on the heterogeneity of a profile, such that a Bergson-Samuelson SWF can be constructed on this domain. The answer is given by showing that a fixed preference profile permits construction of a Bergson-Samuelson SWF if and only if the domain of preferences from which it is derived permits construction of a nondictatorial, neutral Arrow SWF. Thus, the conditions which insure us of the existence of a Bergson-Samuelson SWF are exactly the conditions which insure the existence of an Arrow SWF.

Some notations are necessary at this stage. Let A denote a set of alternatives with at least two elements, let Σ denote the set of all preferences of A, and Ω a subset of Σ. The elements of Ω represent the admissible preference relation in the society. For an integer $n \geq 2$, Ω^n represents the set of all n-tuples of preferences from Ω, and an element of Ω^n, $P=(p_1, p_2, \ldots, p_n) \in \Omega^n$ is an n person profile. An *Arrow social welfare function* (SWF) on Ω is a function $f: \Omega^n \to \Sigma$ which satisfies the following two conditions.

Pareto principle. *For every $P \in \Omega^n$, if xp_iy for all i, then $xf(P)y$.*

Independence of irrelevant alternatives (IIA). *For $x, y \in A$ and $P, Q \in \Omega^n$ if (xp_iy if and only if xq_iy for all i), then ($xf(P)y$ if and only if $xf(Q)y$).*

f is *dictatorial* if there exists an i for which $f(P)=p_i$ for all $P\in\Omega^n$.

For a given fixed profile $P\in\Omega^n$ and $B\subset A$ let $P(B)$ denote the subprofile over this set of alternatives (i.e., the restriction of the preference profile to this set B). For two profiles P and Q and two subsets B and C of A of the same size, we will say that $P(B)=Q(C)$ if there exists a one to one and onto correspondence $f: B\to C$ such that $P(B)=Q(f(B))$. f is *neutral* if for all $P,Q\in\Omega^n$, if $P(x,y)=Q(a,b)$, then $(xf(P)y$ if and only if $af(Q)b)$.[2] A neutral *Bergson–Samuelson social welfare function on P* is a function $f: P\to\Sigma$ which satisfies the following two conditions.

Pareto principle (P*). *If xp_iy for all i, then $xf(P)y$.*

Independence + neutrality (IN*). *For $x,y,z,w\in A$, if $P(x,y)=P(z,w)$, then $(xf(P)y$ if and only if $zf(P)w)$.*

f is *dictatorial* if there exists an i for which $f(P)=p_i$. The relation between the two SWF's will depend on the degree of heterogeneity of the fixed profile P within the restricted domain of preferences Ω.

P is heterogenous in Ω if $P\in\Omega^n$ and for all, $a,b,c\in A$ and $Q\in\Omega^n$, there exists three alternatives $x,y,z\in A$ such that $Q(a,b,c)=P(x,y,z)$.

Thus P is heterogeneous in Ω if the richness in the preferences of Ω is somehow captured in P. Specifically, for all profiles of three alternatives in Ω^n we find three alternatives which exhibit the profile in P. If $\Omega=\Sigma$, i.e., the domain includes all possible preferences over the alternatives of A, then the requirement that P is heterogenous in Ω is equivalent to Axiom U* (unrestricted domain over triples) of Pollak. Thus a corollary of Theorem 2.1 will be Pollak's results which states that when $\Omega=\Sigma$, i.e., preferences are unrestricted, then if P is heterogenous in Σ, there does not exist a nondictatorial Bergson–Samuelson SWF on P.

Theorem 2.1. *Let the fixed profile P be heterogenous in a domain Ω. P permits construction of a nondictatorial neutral Bergson–Samuelson social welfare function if and only if Ω permits construction of a neutral, nondictatorial Arrow social welfare function.*

Proof. Clearly, if f is a neutral, nondictatorial Arrow SWF defined on Ω, then since $P\in\Omega^n$, f is a Bergson–Samuelson SWF on P. To prove that f is nondictatorial, suppose it is not, i.e., there exists j such that $f(P)=p_j$. We want to show that j is an Arrow SWF dictator. Suppose j is not the Arrow SWF dictator, then there exist $Q\in\Omega^n$ and $x,y\in A$ such that xq_jy but $yf(Q)x$. Since P is heterogeneous in Ω, there

[2] A more conventional definition of neutrality for SWF is that any permutation of the names of the alternatives applied to each individual's ordering results in an identical permutation of the names of the alternatives in the social ordering; for SWFs satisfying IIA, the two conditions are equivalent when the domain is unrestricted. The current definition is adopted for the restricted domain case in part because a permuted admissible profile may itself be inadmissible.

exist $a, b \in A$ such that $Q(x, y) = P(a, b)$. Using neutrality of the Arrow SWF we have $f(Q(x, y)) = f(P(a, b))$. Since xq_jy, we have that ap_jb; and since $yf(Q)x$, we have $bf(Q)a$, a contradiction.

As for the reverse, suppose f is a Bergson–Samuelson SWF on a fixed profile P, which is heterogeneous in Ω. To define an Arrow SWF on Ω, let $x, y, z \in A$ and $Q \in \Omega^n$. Since P is derived from Ω, there exist $a, b, c \in A$ such that $Q(x, y, z) = P(a, b, c)$. Define $f(Q(x, y, z)) = f(P(a, b, c))$.

We have to show that f is *well defined, transitive*, satisfies IIA, *Pareto, neutrality*, and is *nondictatorial*.

f is well defined if for all pairs $x, y \in A$, $Q \in \Omega^n$, $xf(Q)y$ or $yf(Q)x$ but not both. Consider a third alternative $z \in A$. Since there exist $a, b, c \in A$ such that $Q(x, y, z) = P(a, b, c)$ and $f(Q(x, y, z)) = f(P(a, b, c))$, either $xf(Q)y$ or $yf(Q)x$. Consider two distinct alternatives z and w, and let $Q(x, y, z) = P(a, b, c)$ and $Q(x, y, w) = P(c, e, g)$. Clearly $P(d, e) = P(a, b)$ and by IN*, $f(P(d, e)) = f(P(a, b)) = f(Q(x, y))$. Thus either $xf(Q)y$ or $yf(Q)x$ but not both.

To prove transitivity, let $x, y, z \in A$, $Q \in \Omega^n$ and $xf(Q)yf(Q)z$. There exist $a, b, c \in A$ such that $Q(x, y, z) = P(a, b, c)$ and $f(Q(x, y, z)) = f(P(a, b, c))$. Therefore $af(P)bf(P)c$. Since f is transitive on P it follows that $af(P)c$ and therefore $xf(Q)z$.

The same proof, mutatis mutandis, shows that IN* and P* imply that f satisfies IIA, neutrality and Pareto. Finally f is nondictatorial since it is nondictatorial on P.

3. On the number of voters

The question of constructing an SWF for any number of voters is discussed in this section. Although a Bergson–Samuelson SWF is defined for a given profile and thus a given number of voters, with additional machinery it is possible to define it for any number of voters.

Given a profile $P \in \Omega^n$, denote by P^k the profile from Ω^k, $k < n$, which is achieved from P by deleting any $n-k$ voters and retaining the order of the remaining k voters.

Proposition 3.1. *Let $P \in \Omega^n$ be heterogeneous in Ω. Then for all $k \leq n$, P^k is heterogenous in Ω.*

Proof. Let $Q \in \Omega^k$. We want to show that for all a, b, c $\exists x, y, z$ such that $Q(a, b, c) = P^k(x, y, z)$. Define $\bar{Q} \in \Omega^n$ by duplicating the kth voter in the last $n-k$ places. Since P is heterogenous in Ω, then $\exists x, y, z$ such that $\bar{Q}(a, b, c) = P(x, y, z)$. Deleting the last $n-k$ voters from $\bar{Q}$ and P yields the desired result.

Given a profile $P \in \Omega^n$ which is heterogeneous in Ω. Let $k \leq n$. We will say that P permits construction of a k voters neutral nondictatorial Bergson–Samuelson SWF if P^k permits construction of such a function.

Theorem 3.2. *Let P be heterogeneous in Ω, for $n \geq 3$. P permits construction of an n person neutral nondictatorial Bergson-Samuelson SWF if and only if it permits construction of a 3 person neutral nondictatorial Bergson-Samuelson SWF.*

Proof. (a) If there exists a three person neutral nondictatorial Bergson-Samuelson SWF on P, then by Theorem 2.1 there exists a 3 person neutral nondictatorial Arrow SWF on Ω since P^3 is heterogeneous in Ω. By adding the rest of the $n-3$ voters as dummies (i.e., their votes do not change the outcome) we get an n person neutral nondictatorial Arrow SWF on Ω, and by Theorem 2.1 there exists an n person nondictatorial neutral Bergson-Samuelson SWF on P.

(b) If there exists an n person neutral nondictatorial Bergson-Samuelson SWF on P, then by Theorem 2.1 there exists an n person neutral nondictatorial Arrow SWF on Ω.

For $n \geq 4$ and $P \in \Omega^n$, let $f(P)$ be an n person neutral nondictatorial Arrow SWF. Define $g_{i,j}$ as follows:

$$g_{i,j}(p_1, p_2, \dots, p_{n-1}) = f(p_1, p_2, \dots, p_{j-1}, p_i, p_j, p_{j+1}, \dots, p_{n-1}).$$

Thus all the $g_{i,j}$'s are $(n-1)$ person Arrow SWF. By a lemma of Kalai and Muller (1977) at least one of these $g_{i,j}$'s is nondictatorial. We wish to show that it is neutral as well.

Let

$$\begin{aligned} &P = p_1, \dots, p_{n-1}, \qquad Q = q_1, \dots, q_{n-1}, \\ &\tilde{P} = p_1, p_2, \dots, p_{j-1}, p_i, p_j, p_{j+1}, \dots, p_{n-1}, \\ &\tilde{Q} = q_1, q_2, \dots, q_{j-1}, q_i, q_j, q_{j+1}, \dots, q_{n-1}. \end{aligned}$$

Since, by assumption, $P(x,y) = Q(a,b)$, it follows that $\tilde{P}(x,y) = \tilde{Q}(a,b)$. Since $g_{ij}(P) = f(\tilde{P})$ and $g_{ij}(Q) = f(\tilde{Q})$, we have the desired result noting that f is neutral. Thus we can define a three person nondictatorial neutral Arrow SWF on Ω. Since P^3 is heterogeneous in Ω, by Theorem 2.1 there exists a 3 person neutral nondictatorial Bergson-Samuelson SWF on P.

The theorem cannot be extended to the 2 person case. To see this, note that there does not exist any 2 person neutral nondictatorial Arrow SWF on any domain. To prove this, suppose there exists such a function. Let $i \neq j$, $i, j \in \{1,2\}$. If voter i is decisive for no pair, voter j is a dictator. Thus voter i is decisive for at least one pair. Using neutrality he is decisive for all pairs. Now let voter i vote xy and voter j vote yx. The result of such a vote is xyx - a contradiction.

4. The nonneutral case

If we discard the neutrality condition Independence + Neutrality, we discard not only neutrality but an independence condition as well. The inherent difficulty in imposing an independence condition is that the condition is a multiprofile one.

However, since the previous theorem assures us that if an n person neutral Bergson-Samuelson SWF exists, then for all $k \leq n$ a k person neutral Bergson-Samuelson SWF exists, we can require the same in the nonneutral case. Then an independence condition is readily available: Let B be a coalition of size k, let $P \mid B$ be the preference profile of the coalition and let $P \mid B(x, y)$ be the restriction of this subprofile to the pair x, y. Let P be a fixed profile. A Bergson-Samuelson SWF on P is a function $f: P \to \Sigma$ which satisfies the following two conditions.

Pareto principle. *If xp_iy for all i, then $xf(P)y$.*

Independence. *Let B and C be two coalitions of size $k \leq n$. Then $P \mid B(x, y) = P \mid C(x, y)$ implies that $xf(P \mid B)y$ if and only if $xf(P \mid C)y$.*

Thus if we take two coalitions of the same size k, since f is defined (as in the previous section) on k voters, if the two (k voters) coalitions vote the same on a pair (x, y), then the function should yield the same ranking for the two alternatives under the votes of the two coalitions.

The heterogeneity condition is applicable to the neutral case but clearly will not be operative for a function which is not neutral.

Consider a profile P which is heterogeneous in some domain Ω. Define $T(\Omega)$ as the set of all triples of Ω, i.e.,

$$T(\Omega) = \{(x, y, z) \in A^3 : \exists q \in \Omega \text{ with } xqyqz\}.$$

If there exists a triple $(x, y, z) \in T(\Omega)$ which does not appear in the given profile P, then clearly there exists a preference q (with $xqyqz$) which does not appear in the profile P. By definition the profile P is heterogeneous in $\Omega' = \Omega - \{q\}$ as well. Given P which is heterogeneous in a domain Ω, we can therefore define Ω^* as the smallest domain in which P is heterogeneous. The fact that it exists is assured by the fact that Ω exists and $P \in \Omega^n$. The domain Ω^* has the property that all its triples appear in the profile P. This condition is used to define heterogeneity in the nonneutral context.

A fixed profile P is *nonneutrally-heterogeneous* in Ω if $P \in \Omega^n$ and for all triples (x, y, z), (a, b, c) if $(x, y, z), (a, b, c) \in T(\Omega)$, then there exist $i < j$ such that $(x, y, z) = p_i$ and $(a, b, c) = p_j$. As in the heterogeneous case, the richness of Ω is captured in P. Here we do not need that all profiles of n voters are captured in P but only two for each two triples. A Bergson-Samuelson SWF f is *nondictatorial* if it is nondictatorial for all $k \leq n$.

Theorem 4.1. *Let the fixed profile P be nonneutrally-heterogeneous in a domain Ω. P permits construction of a nondictatorial Bergson-Samuelson SWF if and only if Ω permits construction of a nondictatotrial Arrow SWF.*

Proof. If f is a nondictatorial Arrow SWF defined on Ω, then define a two person Bergson-Samuelson SWF on P such that it coincides with the two person Arrow

SWF on Ω. The two functions will have the same domain of triples since P is non-neutrally-heterogeneous in Ω. Whether a domain is dictatorial or not depends only on $T(\Omega)$ (see Muller (1982)). Thus there exists at least one nondictatorial 2 person Arrow SWF on Ω. For any $2 < k \leq n$ add the rest of the $k-2$ voters as dummies to achieve a k person nondictatorial Bergson-Samuelson SWF. The same proof mutatis mutandis proves the reverse.

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