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OPTIMAL DYNAMIC DURABILITY*

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Consider a firm that adjusts its production and the choice of durability for its products instantaneously. We show that when the marginal cost with the respect to durability is nonincreasing, (a) the optimal durability for both the competitive firm and the monopolist decreases over time and (b) the monopolist will produce a good with lower durability than the competitive firm. We thus lend support for empirical findings and causal observations that found the phenomenon of declining durability over time.

1. Introduction

Lawrence White (1971) reports that for the U.S. automobile industry since 1945, the *durability* of cars of all makes, for cars of the 'big three', and for each of the main brands *declines over time*. The decline in durability is rather noticeable. For example, the percentage of nine-year-old cars (out of their original production) still on the road has declined from about 80 percent in 1955 to about 55 percent in 1967. The same percentage for eight-year-old cars has declined from 86 percent to 74 percent. The final conclusion remains true even after considering several other possible explanatory variables such as changes in relative prices, disposable income, repair costs, and the like.

Economists have studied the behavior of the product durability over time. Of special interest is the effect of market structure – perfect competition versus monopoly – on the behavior. This extensive work done on durability is limited in some sense. The work has either dealt with the durability in the steady state, or has considered previous time periods but restricted the analysis to the optimal level at the steady state, not considering the periods leading to the steady state. See Kleiman and Ophir (1966), Levhari and Srinivasan (1969), Swan (1970), Levhari and Peles (1973), Kamien and Schwartz (1974), Auernheimer and Saving (1977), and Abel (1983).

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Many goods are produced while the market is not in a steady state. This is especially true for new products entering the market. The interim periods are important since, in many cases, long before the steady state is reached, the market conditions change due to the introduction of new technologies, introduction of new products, and the like.

In this paper we deal with the behavior of durability in dynamic market conditions (mainly new products) in a unified way. The analysis presents a rationale for empirical findings such as White's declining durability.

Our main assumption regarding the issue of durability and market structure is that the marginal cost of production is increasing in production. This contrasts with the constant returns to scale assumption of Swan (1970) that enables him to derive his claimed independence between market structure and durability. Later studies by Sieper and Swan (1973) and by Auernheimer and Saving (1977) indicate that the 'general message ... seems to be that the relaxation of the constant returns assumption does not upset the Swan independence result in the long run' [Schmalensee (1979)].

This work indicates that the constant returns assumption is essential in deriving the independence condition.

The paper is organized as follows: First, we compare the two common types of durable goods discussed in the literature: the 'one-hoss-shay' good that breaks down but never wears out and the exponentially decaying good. We show that the total amount of service a good produces is a measure of durability that is consistent across the decay methods. We then show that, for an continuously decaying good, when the marginal cost of durability does not increase with production, then along the path leading to the steady state (a) both the competitive firm and the monopolist (both seller and renter) will choose an optimal durability path that declines over time and (b) the monopolist will produce a good with lower durability than the competitive firm.

This paper is an extention of the authors' work on the subject [Muller and Peles (1989)] that deals with the one-hoss-shay depreciation method. As will be noted in the next section, although the form of depreciation is called exponential decay, it is, in effect, the most general form of continuous depreciation.

A point of interest is that the paper utilizes the technique of optimal control with integral state equation. This technique is useful whenever the path of the (capital) stock determines future actions and not only the size of the stock. See, for example, Kamien and Muller (1976).

2. Decay of durable goods

There are two types of durable goods discussed in the literature: one that decays exponentially and the other that breaks down abruptly as in the one-hoss-shay case. The firm that uses the services of the good, i.e., the lessee, pays for a unit of service per period and, in the absence of transaction costs, is indifferent between two types of goods. To see the difference it makes on the decision of the producing firm, consider a machine that, when leaving the factory, yields exactly one unit of service for the first period (year). In the one-hoss-shay method, it yields one unit of service per year over its entire finite life time (denoted by N), and then suddenly evaporates and yields zero units of service thereafter. The total amount of service it yields is exactly N over its entire life time. In the exponential decay method, the machine yields one unit of service for the first year and less each year thereafter, at a constant proportional rate.

If the firm decides to add to the durability of this year's vintage of machines, in the one-hoss-shay method, this will not affect the stock of service until the last year of the machine. In the depreciation method, it affects the stock starting next year as less of the stock will depreciate next year since this year's vintage is more durable. Thus the firm's considerations concerning optimal durability are much more complex for the depreciation method since the firm has to consider the price movement for the service from now to the final horizon and not just the price in the last year of the machine.

In terms of the stock of services, however, there is very little difference. The stock depreciates at some rate. The firm can control the stock either by changing its production level or by controlling the depreciation rate. It makes little difference whether it does it by producing machines that last longer but do not depreciate or by producing machines that depreciate more slowly but do not die. To see this, let the good produced decay at a rate of N^{-1}. The total amount of service it renders over its life time is given by

$$\int_0^{\infty} \mathrm{e}^{-N^{-1}t}\,\mathrm{d}t = N.$$

Thus the durability N has exactly the same interpretation as in the one-hoss-shay method. There the product yields one unit of service per year for N years and thus its total service is N. In the depreciation method it yields one unit of service for the first year and depreciates at a rate of N^{-1} thereafter. Its total service over its (infinite) life time is exactly N. The one advantage of exponential decay is that it indirectly takes into account the cost of maintenance. If these increase with age of the product, the net services supplied by the good is decreasing.

Let $Q(t)$ and $x(t)$ be the total stock of services available in the market at time t and the quantity produced at time t, respectively. When the machines produced depreciate at a constant rate N^{-1} (at a steady state, for example), the equation that describes the accumulation of the total quantity is given by

$$\dot{Q}(t) = x(t) - N^{-1}Q(t),$$

where a dot represents differentiation with respect to time.

When the depreciation N^{-1} is not constant, the equation obviously does not hold. Note that, if a function $N(t)$ replaces N in the first equation, this implies that once the firm sets the new durability, and therefore the depreciation for the goods produced today, this will be the rate at which the whole stock decays today. This will be true, for example, with respect to accumulation of nominal wealth. It depreciates at the current inflation rate, regardless of the inflation that existed at the time parts of it were earned.

In a durable stock case, each unit depreciates at the rate which was decided upon when the unit was produced. The total stock is, therefore, the summation of all investments done in the past, where each unit depreciates at its own rate. The equation, thus, is given by

$$Q(t) = \int_0^t \mathrm{e}^{-N^{-1}(s)(t-s)} x(s)\,\mathrm{d}s. \tag{1}$$

This equation is a variant of the one given by Arrow (1964) who first investigated a problem of capital accumulation with variable depreciation; see also Kamien and Muller (1976) and Auernheimer and Saving (1977). In the following two sections, we solve the firm's problem of choosing production rate and durability path so as to maximize its (discounted) profits.

Note that since $N^{-1}(t)$ is not prespecified, the description of eq. (1) is the *most general form* of continuous depreciation. For example, if an arbitrary function $f(t)$ is desired as a depreciation function, this can be achieved by setting $N(t)$ according to

$$N(t) = -1/\lg f(t).$$

3. Optimal durability path

We assume that the firm produces a good which supplies some type of a service. The demand for the good produced is for its service supplied during the same period. That is, the periodic rental price depends on the total quantity Q available in that specific period. Services are proportional to quantity of goods available. We discuss the problem of a seller as well as a firm that rents its products.

Firms have the usual U-shaped average cost curve and production takes place at the rising section of the marginal cost function. Since $Q(0) = 0$ and the quantity at the steady state is a positive, the total quantity is increasing over time. We assume it increases monotonically.[1]

[1]This assumption is related to the assumption we make later on about the existence of an interior solution. In principle, there could be an optimal solution in which $N(t) = 0$ for some values of t, thus reaching a boundary of the feasible set. In this case this monotonicity condition, the local concavity of the Hamiltonian, and the necessary condition (8) need not hold. We do not know, however, an existence theorem for optimal control with integral state equation that will assure us interiority as required.

We assume that firms are operating at decreasing returns to scale and so the steady state is not reached immediately. We also assume that the interest rate remains constant over the entire period.

The decision each firm is facing is the optimal quantity to produce each period (x) and its expected durability (N). For simplicity, but without loss of generality, all goods produced at a given period have the same durability N, yet the durability can (and will) differ between different periods. The relationship of the cost of producing the units x and determining the durability N is given by

$$C(x, N) = x\psi(N)\phi(x) + f(x), \tag{2}$$

i.e.,

$$(1/x)\,\partial C(x, N)/\partial N = \psi'(N)\phi(x), \tag{3}$$

where ϕ is some function which can decrease, increase, or be constant in x.

Given these assumptions, the firm chooses a path of production $x(t)$ and durability $N(t)$ so as to maximize profits given by

$$\pi = \int_0^\infty \mathrm{e}^{-rt}[P(N(t), t)\, x(t) - C(x(t), N(t))]\,\mathrm{d}t, \tag{4}$$

subject to the state eq. (1), where $P(N(t), t)$ is the selling price of the product at time t, when its durability is $N(t)$, and r is the discount rate.

The exact form of the relation between the selling price and the durability can be ascertained via the rental price for the product which is denoted by $p(Q(t))$. The durability of a product produced at time t is fixed at $N(t)$. Thus the services it yields deteriorate at a rate of $N^{-1}(t)$. The selling price will be the discounted stream of rental prices of the service when the discounting takes into account not only the deterioration of the services at a rate of $N^{-1}(t)$ but the discount rate r as well. Thus $P(N(t), t)$ is given by

$$P(N(t), t) = \int_t^\infty p(Q(s))\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\mathrm{d}s. \tag{5}$$

Substituting (5) into (4), changing the order of integration, and using eq. (1) yield

$$\pi = \int_0^\infty \mathrm{e}^{-rt}[p(Q(t))Q(t) - C(x(t), N(t))]\,\mathrm{d}t. \tag{6}$$

In the competitive case we will assume that although the price is not a function of the quantity produced by the firm, it is a function of the total industry's output.

The cost function is given by eq. (2), where $\psi' > 0$, $f' > 0$, $\psi'' \geq 0$, and all other second derivatives are positive. The sign of ϕ' will be defined later on. When $\phi'(x) = 0$, the marginal cost with respect to N per unit of x is independent of the level of x.

The analysis is an optimal control problem with integral state equation. The conditions which are given below are necessary conditions for optimality; see Kamien and Muller (1976) for a proof and a general discussion on these types of problems. Define the Hamiltonian as follows:

$$H = \mathrm{e}^{-rt}\{pQ - x\phi\psi - f\} + \int_0^t \mathrm{e}^{-N^{-1}(t)(s-t)} x(t)\lambda(s)\,\mathrm{d}s. \tag{7}$$

The necessary conditions for optimality are $\partial H/\partial N = 0$, $\partial H/\partial x = 0$, and $\lambda = \partial H/\partial Q(t)$. Since we assume the existence of an interior solution, an additional necessary condition is local concavity of the Hamiltonian, i.e., the Hessian matrix should be negative semidefinite at the optimal choice of control variables. To simplify mathematical notations, let $p(s)(1 + 1/\eta(s))$ and $p(t)(1 + 1/\eta(t))$ denote the expressions $p(Q(s))(1 + 1/\eta(Q(s))$ and $p(Q(t)(1 + 1/\eta(Q(t))$, respectively. Solving and substituting for λ yields

$$\psi'\phi = N^{-2}\int_t^\infty (s-t)p(s)(1 + 1/\eta(s))\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\,\mathrm{d}s, \tag{8}$$

where η is the price elasticity of demand,

$$(\phi + x\phi')\psi + f' = \int_t^\infty p(s)(1 + 1/\eta(s))\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\,\mathrm{d}s. \tag{9}$$

Differentiation of eqs. (8) and (9) with respect to time yields the following two equations with two unknown $\dot{x}$ and $\dot{N}$:

$$\begin{aligned}
&\partial^2 H/\partial N^2 \dot{N} + \partial^2 H/\partial N \partial x \dot{x} \\
&= N^{-2} x\,\mathrm{e}^{-rt}\int_t^\infty \left[1 - \left(r + N^{-1}(t)\right)(s-t)\right] \\
&\quad \times p(s)(1 + 1/\eta(s))\mathrm{e}^{-(r+N^{-1})(s-t)}\,\mathrm{d}s,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\partial^2 H/\partial N \partial x \dot{N} + \partial^2 H/\partial x^2 \dot{x} \\
&= -\mathrm{e}^{-rt}\int_t^\infty (r + N^{-1})p(s)(1 + 1/\eta(s))\mathrm{e}^{-(r+N^{-1})(s-t)}\,\mathrm{d}s \\
&\quad + p(t)(1 + 1/\eta(t))\mathrm{e}^{-rt}.
\end{aligned} \tag{11}$$

In appendix 1 we show that the right-hand side of eq. (10) is positive and the right-hand side of eq. (11) is positive, as long as our assumption that the total stock does not decrease over time holds. Using this fact and solving for $\dot{x}$ and $\dot{N}$ yields that, as long as $\phi' \leq 0$ (and so $\partial^2 H/\partial N \partial x \geq 0$), $\dot{x}$ is negative and $\dot{N}$ is negative. This is achieved by noting that the Hessian matrix of the Hamiltonian is negative semidefinite. In addition, if $\psi'' > 0$, then clearly $\partial^2 H/\partial N^2 < 0$ as required by the fact that the Hessian is N.S.D. If, however, $\psi'' \geq 0$, then we show in appendix 2 that $\partial^2 H/\partial N^2$ is still negative.

Thus, the results holds both when there are decreasing returns to scale with respect to durability and constant returns to scale, that is, $\phi' \leq 0$, and production decreases over time as can be expected. Note that production decreases also in the standard case where the firm does not have control over the durability and the latter remains fixed.

In addition, durability decreases over time. The reason behind it relies on the fact that the price (or marginal revenue) declines over time. Thus, when considering the choice of durability now or at a later date, since the discounted (marginal) benefits decline with time, so does the durability.

In case that the marginal cost with respect to durability increases in x, i.e., $\phi' > 0$, durability can either decrease or increase. It depends on the rate at which the price declines, which tends to lower durability, versus the rate at which the marginal cost increases in x (the magnitude of ϕ') which tends to increase durability.

4. Market structure and durability

The analysis so far was performed on the assumption of monopoly. When the setting is competitive, the analysis still holds where the price p replaces the marginal revenue term $p(1+1/\eta)$. Although the firm acts as if the price is given, price does go down to the steady state since it is a function of the total industry's stock on the market.

Although the pattern of behavior of the durability over time towards the steady state of a competitive firm is similar to that of the monopolist, the two functions will be different as a result of the two market structures. To make the comparison meaningful the monopoly and the competitive market structures will produce under similar terms. Thus we assume that the monopoly is a multiplant monopoly, or a cartel acting as a monopolist, such that each plant will operate approximately as a firm in a competitive market. In appendix 3 we show that assuming $\phi' \leq 0$, namely that marginal cost with respect to durability does not increase with production, then both quantity produced x and durability N are increasing functions of the elasticity of demand facing each firm. Thus, if we have two markets, 1 and 2, and $|\eta_1| > |\eta_2|$, then $x_1 > x_2$ and $N_1 > N_2$. A firm whose demand is more elastic will produce more. Since we assume a production function such that with lower output the marginal cost of durability is higher, durability and quantity are positively correlated. Thus the firm with the more elastic demand will produce more and its products will be more durable.

A firm in a competitive market faces an elasticity of demand that is higher than its monopolist counterpart. In fact, it has an infinite elasticity of demand. Thus we obtain that, given our assumption that $\phi' \leq 0$, *the durability of goods produced by a monopoly falls short of the durability of goods that would have been produced by a competitive firm*. Note that, in the case when $\phi' = 0$, only one of Swan's assumptions (of constant returns) is relaxed. The separability between cost of the production and durability is still assumed in this case. This clearly indicates the sensitivity of Swan's independence result to his assumptions.

5. Conclusion

To conclude, we have shown that, when the marginal cost with respect to durability (per unit of production) does not increase with production (i.e., $\phi' \leq 0$), then both competitive firm and a monopoly will choose an optimal durability path which decreases over time. In addition the same condition causes the monopolist to choose a durability which is lower than the durability chosen by the competitive firm along the path leading to this steady state.

Finally, it might be of interest to find out whether the selling price is decreasing through time. To do so, differentiate (5) with respect to time and use the fact that

$$\int_t^{\infty} \mathrm{e}^{-(r+N^{-1}(t))(s-t)} \mathrm{d}s = 1/\left(r + N^{-1}(t)\right),$$

to arrive at the following:

$$\mathrm{d}P/\mathrm{d}t = (\partial P/\partial N)(\mathrm{d}N/\mathrm{d}t) + \left(r + N^{-1}(t)\right)$$

$$\times \int_t^{\infty} [p(Q(s)) - p(Q(t))] \mathrm{e}^{-(r+N^{-1}(t))(s-t)} \mathrm{d}s.$$

Since the rental price declines through time, the second term is negative. The first term is negative if durability declines through time since using (5) $\partial P/\partial N$ is clearly positive. Thus, if the marginal cost of durability does not increase in production, the selling price declines through time. If the reverse holds, however, there might be a case where durability increases at such a rate that, although the rental price declines, the selling price increases because of the higher durability.

Appendix 1

In order to prove that

$$\int_t^{\infty}\left[1-\left(r+N^{-1}(t)\right)(s-t)\right] p(s)\left(1+1/\eta(s)\right) \mathrm{e}^{-(r+N^{-1}(t))(s-t)}\mathrm{d}s$$

is positive, define s^* as the unique s that solves the equation $(r+N^{-1}(t))\cdot(s^*-t)=1$, that is, $s^*=t+1/(r+N^{-1}(t))$. In addition define $p^*(1+1/\eta^*)=p(s^*)(1+1/\eta(s^*))$. Since $p(1+1/\eta)$ declines over time, the following inequalities hold:

$$1-(r+N^{-1})(s-t)\geq 0,$$

$$(1+1/\eta(s))p(s)\geq(1+1/\eta^*)p^* \quad \text{for} \quad t\leq s\leq s^*,$$

and

$$1-(r+N^{-1})(s-t)<0,$$

$$(1+1/\eta(s))p(s)<(1+1/\eta^*)p^* \quad \text{for} \quad s>s^*.$$

Therefore the following inequality holds:

$$\int_t^{\infty}\left[1-\left(r+N^{-1}(t)\right)(s-t)\right] p(s)\left(1+1/\eta(s)\right)\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\mathrm{d}s$$

$$>p^*(1+1/\eta^*)\int_t^{\infty}\left[1-\left(r+N^{-1}(t)\right)(s-t)\right]\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\mathrm{d}s\geq 0,$$

where the last inequality is achieved by performing the integration. In order to show that the expression

$$-\int_t^{\infty}\left(r+N^{-1}(t)\right)p(s)(1+1/\eta(s))\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\mathrm{d}s$$

$$+p(t)(1+1/\eta(t))$$

is positive, note that, since $p(1+1/\eta)$ declines with time, the above expression is strictly larger than the following:

$$p(t)(1+1/\eta(t))\left[-\left(r+N^{-1}(t)\right)\int_t^{\infty}\mathrm{e}^{-(r+N^{-1}(t))(s-t)}\mathrm{d}s+1\right]=0,$$

where the last equality is achieved by performing the integration.

Appendix 2

In order to show that, even when $\psi'' = 0$, $\partial^2 H/\partial N^2 < 0$, we have to show the positivity of the following expression:

$$\int_t^\infty (2N(t) - (s-t))(s-t)\mathrm{e}^{-(r+N^{-1}(t))(s-t)} p(s)(1 + 1/\eta(s)\,\mathrm{d}s.$$

Define s^* as $s^* = t + 2N$. As in appendix 1, since $p(1.+1/\eta)$ declines over time, this expression is larger than the following expression:

$$p^*(1+1/\eta^*)\int_t^\infty (2N - (s-t))(s-t)\mathrm{e}^{-(r+N^{-1})(s-t)}\mathrm{d}s.$$

Performing integration by parts yields that the last expression is positive.

Appendix 3

In this appendix we show that both production and durability are increasing functions of the absolute value of the elasticity of demand. Thus, since η is negative, we have either $\partial x/\partial\eta < 0$, $\partial N/\partial\eta < 0$ or $\partial x/\partial|\eta| > 0$, $\partial N/\partial|\eta| > 0$.

Differentiation of (8) and (9) with respect to η yields the following two equations with two unknown $\partial x/\partial\eta$ and $\partial N/\partial\eta$:

$$\partial^2 H/\partial N^2 \partial N/\partial\eta + \partial^2 H/\partial N \partial x\, \partial x/\partial\eta$$

$$= N^{-2}\int_t^\infty (s-t)P\eta^{-2}\mathrm{e}^{-(r+N^{-1})(s-t)}\mathrm{d}s,$$

$$\partial^2 H/\partial N\partial x\,\partial N/\partial\eta + \partial^2 H/\partial x^2 \partial x/\partial\eta = \int_t^\infty P\eta^{-2}\mathrm{e}^{-(r+N^{-1})(s-t)}\mathrm{d}s.$$

As in the previous appendix, since the Hessian matrix of the Hamiltonian is negative semidefinite, the left-hand side of the above two equations is positive, and $\partial^2 H/\partial N\partial x > 0$ (since $\phi' \le 0$), we have that both $\partial x/\partial\eta$ and $\partial N/\partial\eta$ are negative.

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