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The Equivalence of Strong Positive Association and Strategy-proofness

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1. INTRODUCTION

Consider a group that must select one alternative from a set of three or more alternatives. Each member casts a ballot that the voting procedure counts. For a given alternative x, let two ballot profiles C and D have the property that if a member ranks alternative x above alternative y within C, then he also ranks x above that y within D. Strong positive association requires that if the voting procedure selects x when the profile is C, then it must also select x when the profile is D. We prove that strong positive association is equivalent to strategy-proofness. It therefore follows that no voting procedure exists that satisfies strong positive association, nondictatorship, and citizens' sovereignty.

Define a *group* to be a set N whose $|N|$ elements are the group's *members*. They select one element from the *set of alternatives*, S, by each casting a ballot and then using a voting procedure to count the ballots. A *ballot* B_i is a strict ordering of the elements within S, e.g., $B_i = (xyz)$ where $S = \{x, y, z\}$ and x is ranked highest, y second highest, and z lowest. Indifference is not allowed. A *voting procedure* is a single-valued function $v(B_1, ..., B_n)$ that evaluates the profile of ballots and selects one element of S as the group's chosen alternative.

Each member $i \in N$ has preferences P_i over the set of alternatives S. *Preferences*, like a ballot, are a complete, asymmetric, and transitive ordering of S. A member's preferences P_i describe what he truly desires. For example, $P_i = (xyz)$ denotes that individual i most prefers that the group's choice be x, next prefers that it be y, and least prefers that it be z. An alternative notation for the preference ordering $P_i = (xyz)$ is xP_iy, xP_iz, and yP_iz, where xP_iy means individual i prefers x to y. Similarly an alternative notation for the ballot $B_i = (xyz)$ is xB_iy, etc. Beyond completeness, asymmetry, and transitivity we place no restrictions, such as single-peakedness, on either

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admissible preferences or admissible ballots. Any strict ordering is admissible. Indifference, however, is excluded as inadmissible.

A group member's choice of ballot B_i need not be identical to his true preferences P_i. If a member selects a ballot B_i which is identical to his preferences P_i, then $B_i \equiv P_i$ is called his *sincere strategy*. If, however, he selects a ballot B_i which is different than his preferences, then $B_i \not\equiv P_i$ is called his *insincere strategy*. The n-tuple of sincere strategies $B \equiv (B_1, ..., B_n) \equiv (P_1, ..., P_n) \equiv P$ is called the *sincere strategy profile.*

Consider two related ballot profiles $B = (B_1, ..., B_n)$ and $B' = (B_1', ..., B_n')$. Suppose that $v(B_1, ..., B_n) = x$. Suppose also, that for all members $i \in N$ and all alternatives $y \in S$, if alternative x is ranked above alternative y on ballot B_i, then on ballot B_i' alternative x is also ranked above alternative y. In other words, the switch from profile B to profile B' precludes any alternative y, which was ranked below alternative x on ballot B_i, from jumping above alternative x on ballot B_i'. Given these ballot profiles B and B', a reasonable requirement to place on the manner in which v counts the ballots is that $v(B_1', ..., B_n') = x$. After all, $v(B_1, ..., B_n) = x$ and, in the switch from profile B to profile B', alternative x has retained or improved its relative position with respect to every other alternative. We call this requirement, which will be defined more formally shortly, strong positive association.

An example where $S = \{w, x, y, z\}$ and $|N| = 5$ shows that both plurality rule and the Borda count fail to satisfy this requirement. Let profiles B and B' be:

$$\begin{aligned} B_1 &= (xwzy), & B_1' &= (xyzw), \\ B_2 &= (xwzy), & B_2' &= (xzyw), \\ B_3 &= (yzxw), & B_3' &= (yzxw), \\ B_4 &= (zyxw), & B_4' &= (yzxw), \\ B_5 &= (wyxz), & B_5' &= (yxwz). \end{aligned}$$

With respect to alternative x, profiles B and B' satisfy the requirements of strong positive association. Note that the condition's requirements on B and B' relate only to those pairs of alternatives that include x, not to all possible pairs of alternatives. Consider plurality rule (v_p) first. Profile B gives x two first place votes, compared to one first place vote each for the other elements of S. Therefore $v_p(B) = x$. But contrary to the requirement of strong positive association, $v_p(B') = y$. The story repeats itself for the Borda count (v_B).[1] For profile B alternative x receives 9 points while the other

[1] The Borda count, which is named after its eighteenth century French inventor, selects a winning alternative by assigning each alternative $(|S| - k - 1)$ points for each ballot in which it is ranked k positions from the top. The points for each alternative are summed and the winner is that alternative which receives the most points. If two alternatives receive the same number of points, then individual one's ballot, B_1, is used to break the tie.

alternatives each receive 7 points. For profile B', alternative y receives 12 points while alternatives w, x, and z, respectively, receive 1, 10 and 7 points. Therefore $v_B(B) = x$ and $v_B(B') = y$.

2. Equivalence of Strong Positive Association and Strategy-proofness

The formal definitions of the two conditions are:

Strong positive association (SPA).[2] For any $x \in S$, let C and D be any two ballot profiles such that, for all $s \in S$ and all $i \in N$, xC_is implies xD_is. A voting procedure satisfies SPA if and only if $v(C) = x$ implies $v(D) = x$.

Strategy-proofness (SP). A voting procedure satisfies SP if and only if no sincere strategy profile $C \equiv P$ exists such that, for some member $i \in N$ and for some insincere strategy C_i',

$$v(C/C_i')\, P_i v(C/C_i), \tag{1}$$

where

$$C/C_i' = (C_1, ..., C_{i-1}, C_i', C_{i+1}, ..., C_n) = P/C_i' \tag{2}$$

and

$$C/C_i = (C_1, ..., C_{i-1}, C_i, C_{i+1}, ..., C_n) = P/P_i = P. \tag{3}$$

Strategy-proofness requires that no member has an incentive to employ an insincere strategy. Specifically, notice in (1) that $C_i \equiv P_i$ because C_i is the sincere strategy of member i; therefore (1) may be rewritten as $v(C/C_i')\, C_i v(C/C_i)$.

Theorem. *A voting procedure satisfies* SP *if and only if it satisfies* SPA.[3]

Proof. First we prove that SP implies SPA. Suppose that SP does not imply SPA. Therefore a voting procedure v exists that satisfies SP, but not

[2] We have named this condition "strong positive association" because it is a straightforward strengthening of Arrow's condition of "positive association" (see [1, p. 26]). One can show that SPA implies positive association. In addition, one can show that if, in our definition of a voting procedure, we permitted the set S of feasible alternatives to vary over some universal set of alternatives, then SPA also implies Arrow's independence of irrelevant alternatives condition. The converse, however, is not true. For further discussion of the relationship between SP (and SPA) and Arrow's condition see [2] and [6].

[3] If we had not assumed that every strict ordering over S is admissible as members' preferences and ballots, then the theorem would hold in one direction only. Inspection of the proof shows that if the set of admissible preferences and ballots is restricted, then SP implies SPA while SPA does not necessarily imply SP.

SPA. This means that distinct alternatives x, $z \in S$ and profiles B and C must exist such that, for all $i \in N$ and all $y \in S$,

$$xB_i y \Rightarrow xC_i y, \tag{4}$$

$v(B) = x$, and $v(C) = z$. Consider the sequence

$$\begin{aligned}
&v(B_1, B_2, \ldots, B_n) = x, \\
&v(C_1, B_2, \ldots, B_n), \\
&\qquad\cdots \\
&v(C_1, \ldots, C_{j-1}, B_j, B_{j+1}, \ldots, B_n) = v(D/B_j), \\
&v(C_1, \ldots, C_{j-1}, C_j, B_{j+1}, \ldots, B_n) = v(D/C_j), \\
&\qquad\cdots \\
&v(C_1, \ldots, C_{n-1}, B_n), \\
&v(C_1, \ldots, C_{n-1}, C_n) = z.
\end{aligned} \tag{5}$$

Since $v(B) = x$ and $v(C) = z$, a switching point must exist; a $j \in N$ exists such that $v(D/B_j) = x$ and $v(D/C_j) = u$ (u might equal z). Two possibilities exist for B_j, either xB_ju or uB_jx, and both lead to contradictions. If xB_ju, then (4) implies that xC_ju. Therefore $v(D/B_j)\, C_j v(D/C_j)$, which is a contradiction of SP. If uB_jx, then $v(D/C_j)\, B_j v(D/B_j)$, which is also a contradiction of SP. Therefore SP implies SPA.

We now show that SPA implies SP. Suppose that SPA does not imply SP. Therefore a voting procedure v must exist that satisfies SPA but not SP. Consequently a distinct pair x, $y \in S$, a sincere strategy $B_j \equiv P_j$, a profile B, a member $j \in N$, and an insincere strategy B_j' must exist such that

$$v(B/B_j') = y, \tag{6}$$

$$v(B/B_j) = x, \tag{7}$$

and

$$yB_jx. \tag{8}$$

Partition S into three exhaustive and disjoint subsets:

$$W^+ = \{z \in S \mid zB_jx\}; \tag{9}$$

$$X = \{z \in S \mid (xB_jz \,\&\, zB_j'x) \text{ or } z = x\}; \tag{10}$$

$$W^- = \{z \in S \mid xB_jz \,\&\, xB_j'z\}. \tag{11}$$

Construct a ballot Q_j from ballots B_j and B_j' as follows:

$$[s \in W^+ \,\&\, (t \in X \text{ or } t \in W^-)] \Rightarrow sQ_jt; \tag{12}$$

$$(s \in X \,\&\, t \in W^-) \Rightarrow sQ_jt; \tag{13}$$

$$s, t \in W^+ \Rightarrow (sQ_jt \Leftrightarrow sB_jt); \tag{14}$$

$$s, t \in X \Rightarrow (sQ_jt \Leftrightarrow sB_j't); \tag{15}$$

and

$$s, t \in W^- \Rightarrow (sQ_jt \Leftrightarrow sB_jt). \tag{16}$$

The effect of this construction is to order the three sets W^+, X, and W^- in descending order: $Q_j = (W^+XW^-)$. The individual alternatives within W^+ are ordered as ballot B_j ordered them, the alternatives within X are ordered as ballot B_j' ordered them, and the alternatives within W^- are ordered as ballot B_j ordered them. Notice that (10) and (14) imply that if $s \in X$ and $s \neq x$, then sQ_jx.

Denote by w the group choice that the ballot profile B/Q_j generates: $v(B/Q_j) = w$. Since W^+, X, and W^- partition S, three possibilities exist for w: $w \in W^+$, $w \in X$, or $w \in X^-$. We consider each of these possibilities in turn and show that application of SPA, which v is assumed to satisfy, leads to a contradiction of our assumptions (6), (7), or (8). Therefore SPA implies SP.

If $w \in W^+$, then for all $z \in S$, (12) and (14) imply that

$$wQ_jz \Rightarrow wB_jz. \tag{17}$$

Moreover, $v(B/Q_j) = w$, and the only difference between ballot profile B/Q_j and B/B_j is the ballot of member j. Therefore SPA is applicable to the switch from profile B/Q_j to B/B_j. It implies that $v(B/B_j) = w$ because $v(B/Q_j) = w$. But our assumption (7) states that $v(B/B_j) = x$. Since $x \notin W^+$ we have our first contradiction.

If $w \in X$, then for all $z \in S$, (10), (11), (13), and (15) imply that

$$wQ_jz \Rightarrow wB_j'z. \tag{18}$$

The same argument, *mutatis mutandis*, as we used in analyzing the first possibility implies that SPA is applicable to the switch from profile B/Q_j to profile B/B_j'. Therefore, $v(B/B_j') = w$ because $v(B/Q_j) = w$. Assumption (6), however, is that $v(B/B_j') = y$. Moreover, (6) when coupled with (9) implies that $y \in W^+$. Therefore, a contradiction exists because $w \in X$ and y equals w.

The one remaining possibility is $w \in W^-$. If $w \in W^-$, then for all $z \in S$, (12), (13), and (16) imply that

$$wQ_jz \Rightarrow wB_jz. \tag{19}$$

The same argument, *mutatis mutandis*, implies that SPA is applicable to the switch from profile B/Q_j to profile B/B_j. Therefore $v(B/B_j) = w$ because $v(B/Q_j) = w$. But assumption (7) states that $v(B/B_j) = x$. Since $x \notin W^-$ we have our third contradiction. ∎

A corollary to the equivalence theorem is that if the number of alternatives is at least three, then no voting procedure exists that satisfies SPA, citizens' sovereignty, and nondictatorship. Citizens' sovereignty requires that the group can actually choose any alternative within S by casting an appropriate ballot profile. A dictatorial voting procedure vests all decision making power into one individual, the dictator. Formally these two conditions and the corollary are:

Citizens' sovereignty (CS). A voting procedure satisfies CS if and only if, for every alternative $x \in S$, a ballot profile B exists such $v(B) = x$.

Dictatoriality (D). A voting procedure is dictatorial if and only if a member $i \in N$ exists such that, for all ballot profiles B and all alternatives $x \in S$, either $v(B) = x$ or $v(B)\, B_i x$.

COROLLARY. *If S contains at least three alternatives, then every voting procedure that satisfies* SPA *and* CS *is dictatorial.*

The corollary's proof follows directly from the equivalence between SPA and SP and a theorem of Gibbard [3] and Satterthwaite [6]: If S contains at least three alternatives, then every voting procedure that satisfies SP and CS is dictatorial.[4]

3. DISCUSSION

Pattanaik [5] in a recent paper examined the relationship between strategy-proofness and conditions of the "independence of irrelevant alternatives" type. He assumed that voting procedures may select as the group's choice a lottery among several alternatives. This contrasts with our assumption that a voting procedure deterministically selects as the group's choice a single alternative. Within his framework Pattanaik defined several independence conditions, the strongest of which is IC1. He then showed that IC1 is a necessary, but not sufficient, condition for SP. Therefore from our equivalence theorem between SP and SPA it follows that within the context of deterministic voting procedures SPA implies IC1, but not the converse.

Gibbard [4] has also considered, in a somewhat different manner than

[4] It is easily shown that, in conjunction with SP (or SPA), CS and Pareto optimality are equivalent. Therefore the corollary may be stated with Pareto optimality replacing CS.

Pattanaik, voting procedures that admit lotteries as outcomes. He showed [4, Lemma 2] that such voting procedures are SP if and only if they are "localized" and "nonperverse." This result, together with our equivalence result for SP and SPA, imply that, for the case of deterministic voting procedures, his two conditions of localization and nonperversity are jointly equivalent to our single condition of SPA.

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