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Essential Aggregation Procedures on Restricted Domains of Preferences*

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The restricted domains of individuals' preferences that permit the construction of Arrow social welfare functions and nonmanipulable voting procedures in which each of n voters has some power are characterized. In this context a domain is the Cartesian product of n sets of strict preference orderings. Variants of this result are obtained under the additional requirement of neutrality and in the case when alternatives are vectors whose ith components affect only the ith voter. Kalai and Muller's analogous result (*J. Econ. Theory* **16** (1977), 457–469) concerning nondictatorial procedures is discussed and proved as a corollary to the main theorem. *Journal of Economic Literature* Classification Number: 025.

1. INTRODUCTION

We characterize in this paper the domains of individuals' preferences that permit the construction of *essential* aggregation procedures, that is, ones in which each of n voters has some power. We consider two classes of such mechanisms, Arrow social welfare functions (SWFs) and nonmanipulable voting procedures. A domain for a member of one of these classes is the Cartesian product of n sets of preference orderings. An *Arrow SWF* is a function from such a domain to the set of orderings that satisfies the weak

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Pareto condition and independence of irrelevant alternatives. A *voting procedure* associates with each set of n admissible orderings and subset of alternatives a member of the subset. A voting procedure is *nonmanipulable* if sincere revelation of preferences is a dominant strategy for all voters.

It is well known that if no restriction is imposed on the set of admissible orderings, then the only members of these classes are dictatorial, as Arrow [1], Gibbard [3], and Satterthwaite [12] have shown.

Attempts to resolve the difficulties posed by these impossibility theorems through the restriction of admissible preferences have taken two forms. The first is to choose a particular aggregation mechanism (usually majority rule) and then to look for domain limitations sufficient to make that mechanism "well behaved," e.g., transitive or nonmanipulable. Among the outstanding contributions to this strand of literature have been [2, 16, 17]. As Kramer [9] has shown, the necessary and sufficient conditions for the transitivity of majority rule are extraordinary restrictive in the context of the usual economic assumptions of quasi-concavity and differentiability.

The second line of inquiry, pursued by Maskin [10] and Kalai and Muller [4], reverses the procedure. It begins with a fixed domain of admissible preferences and looks for a mechanism that will consistently aggregate preference profiles drawn from this domain. Our arguments are more closely related to those of Kalai and Muller.

The essence of the inverse relationship between the richness of the domain of permissible preferences and the extent of the class of transitive or nonmanipulable procedures lies in what has been called "the contagiousness of decisiveness." (The epidemiological metaphor is Sen's [15, p. 220].) It is easily illustrated for the case of Arrow SWFs. Suppose that some coalition C is known to be *almost decisive* for some alternative x against y under some SWF: that is, whenever all members of C prefer x to y and all others prefer y to x, then x is socially preferred to y. Suppose also that the two orderings $(x \succ y \succ z)$ and $(y \succ z \succ x)$ are permissible preferences for all voters. Now suppose that each member of C has the former ordering and everyone else has the latter. Then by C's almost-decisiveness x is socially preferred to y; by the Pareto condition, y is socially preferred to z. Hence by transitivity x is socially preferred to z. But this means, in view of independence, that C is *also* almost decisive for x against z. All of this is surely familiar to anyone who has read the standard proofs of Arrow's impossibility theorem. The importance of the domain of admissible preferences can be seen by substituting for the second ordering the ordering $(z \succ y \succ x)$; no conclusions about C's almost-decisiveness for other pairs can then be deduced. So clearly some combinations of admissible preferences cause decisiveness to be contagious while others do not. Two types of decisiveness contagion can occur; one is the spread of a coalition's decisiveness from one pair of alternatives to another, while the other is the spread of the decisiveness of one coalition over some pair to that of some other coalition over some other pair. Both varieties appear in most proofs of Arrow's result. From this perspective, that theorem is the demonstration that, if no domain restrictions are imposed, then the existence of one decisive set (guaranteed by the weak Pareto condition) implies that decisiveness is quite pandemic; indeed, there then exists a single-membered coalition that is decisive over all pairs—a dictator.

Kalai and Muller have considered the problem of constructing aggregation procedures with the weakest possible axiom regulating the distribution of voting power: that there be no dictator. Kalai and Muller require that the same orderings be permissible for all individuals. Under this common-domain assumption, they have given a characterization of the admissible sets of preferences for individuals that will permit the set of ordered pairs of distinct alternatives to be partitioned into two nonempty subsets with a particular set of properties. The partition depends, in general, on the admissible preferences. The first property is that, given the fixed domain, if a coalition were to be decisive for a pair belonging to one subset, then that coalition could not be shown to be decisive for a pair belonging to the other subset by an argument like the one given earlier. The second property is that, given the fixed domain, if a coalition were to be decisive for a pair belonging to one of the subsets, then *every* pair over which that coalition could be shown to be decisive by such an argument must also belong to that same subset. If such a partition exists, the set of pairs of alternatives is said to be *decomposable* with respect to the domain. If the set of pairs is decomposable, then it is clearly feasible to "quarantine" two distinct epidemics of decisiveness.

Kalai and Muller then show that a SWF can be constructed on any domain that permits such a decomposition of the set of pairs in the following way: let one voter be decisive over the pairs in one subset, let another voter be decisive over the pairs in the other subset, and let all other voters be dummies, that is, individuals whose preferences never count. On such a restricted domain this aggregation procedure satisfies all of Arrow's axioms including, obviously, nondictatorship. Decomposability with respect to a domain is also necessary and sufficient, they show, for the existence of a nondictatorial, nonmanipulable voting procedure on that domain. Thus they extend Satterthwaite's equivalence theorem [12] to the case of restricted domains.

Maskin [10] and Kalai and Muller [4] worked with the two-person case because the existence of a nondictatorial SWF for a particular domain of preferences is independent of the number of voters. That is, there exists an n-person SWF on a domain if and only if there exists a two-person SWF on the same domain. The characterization and correspondence theorems were proved, therefore, for the simple two-person case. Their result on the irrelevance of the size of the electorate, however, does not hold for essential SWFs. As we shall show in Section 3, for arbitrary positive integers m, there exist domains for which an essential SWF can be found if there are m individuals or fewer, but not if there are more. This dependence complicates our domain characterization task.

The Kalai–Muller procedures are certainly nondictatorial; it is equally certain, however, that no collectivity larger than two persons would voluntarily adopt such a decision mechanism. However appealing nondictatorship is as a necessary condition for an aggregation procedure,[1] it is hardly an adequate sufficient specification of the desired distribution of power. In this paper we pose and answer the Kalai–Muller question for some classes of somewhat more egalitarian procedures. We first require that no voter be inessential, that is, completely deprived of power. In this context, a person is *essential* under some aggregation procedure iff there exists a set of admissible voters' preferences such that an admissible change in that person's preferences would alter the value of that procedure. An analogous notion of decomposability with respect to a domain will be defined for this case in Section 3. We show in Theorem 1 that this condition is necessary and sufficient for the existence of essential Arrow SWFs. Strengthening nondictatorship to the requirement that all voters are essential has a nontrivial effect on the set of domains for which well-behaved procedures exist. Nondictatorship is equivalent to essentiality only in the two-person case, and then only when both individuals have identical sets of admissible orderings.

Our framework is more general than Kalai and Muller's in a second important way. Their theorem precludes admissible orderings from differing across individuals. In some circumstances it may be appropriate to model different social groups as having their own sets of admissible preferences. We consider the most general of these situations, in which each individual can have a distinct set of admissible orderings. Economic models often impose more structure on the set of alternatives than is typically assumed in pure social choice theory. In such cases this additional structure may suggest restrictions on the ways in which economic policy alternatives or allocations may affect different individuals, and thus delimit their sets of conceivable preferences over the alternatives.

We also demonstrate that the domain restriction necessary and sufficient for the existence of an n-person essential SWF (satisfying independence of irrelevant alternatives, monotonicity, and the weak Pareto condition) is identical to the restriction guaranteeing the existence of an n-person nonmanipulable essential voting procedure.

[1] Nondictatorship is not always an appealing necessary condition when domains are restricted. If all individuals share a fixed admissible ordering, every procedure satisfying the Pareto condition makes *everyone* a dictator.

Since we require that every individual have some power, it is natural to explore the possibility of manipulation of a voting procedure by a coalition. That is, can a group of voters coordinate its preference misrepresentations to manipulate the collective outcome in its favor? We show that groups can manipulate if and only if some individual can manipulate. Thus the notions of individual manipulability and group manipulability of a voting procedure are completely equivalent in this context.

Our main result therefore establishes the triple equivalence of

(1) the existence of an n-person essential SWF on a domain,

(2) the existence of an n-person essential voting procedure on that domain that is immune to manipulation by groups, and

(3) the decomposability of the set of pairs with respect to that domain.

Kalai and Ritz [5] have proved a variant of the Kalai–Muller theorem for one specification of additional structure on the feasible alternatives. They assume that the objects of choice are n-dimensional vectors, with the ith component of such a vector representing the part of the alternative affecting only the ith individual. For example, such a vector might represent an allocation of a single commodity in an economy without consumption externalities. In Section 4 we prove an analogue to the Kalai–Ritz result. We replace in their theorem nondictatorship with essentiality for arbitrary numbers of individuals and we allow admissible orderings over components to differ across individuals.

Section 4 also provides an alternative proof of the Kalai–Muller theorem by showing the equivalence, in a common-individual-domain context, of the Kalai–Muller decomposability condition and essential decomposability for two persons, as well as the equivalence of nondictatorship and two-person essentiality.

We may well wish to tighten further our requirements on admissible aggregation procedures since essentiality does not prohibit wide disparities in voting power. One additional axiom that increases the minimum ability of voters to affect the outcome (in the presence of essentiality) is *neutrality*, the requirement that an aggregation procedure treat alternatives symmetrically. Of course, other reasons may be advanced for imposing this condition, such as the simplicity of neutral procedures. For the case in which individuals have identical sets of admissible preferences, we give in Section 5 a characterization of the domains on which well-behaved neutral essential procedures exist. This result is of interest in part because the domain restriction for this case is much more transparent than any of the other decomposability conditions in this paper or elsewhere in the literature.

Section 6 contains some concluding remarks, including a discussion of the likelihood that the decomposability conditions will be satisfied.

2. Notation and Definitions

This section introduces the terminology and notation needed in Section 3. Let A denote a set of mutually exclusive social alternatives or candidates. The set N consists of the set of individuals or voters; it has n members. Let P_i and Q_i, $i = 1,..., n$, denote individuals' preferences, which are assumed throughout the paper to be strong orderings. Each individual is assumed to have an admissible set of preference orderings Ω_i. A *domain* $\Pi\Omega_i$ is a set of n-tuples of admissible orderings. The domains we consider will generally be proper subsets of the *unrestricted domain* U^n, where U is the set of all possible strong orderings of A. A *profile* $P = (P_1,..., P_n)$ is an element of a domain $\Pi\Omega_i$.

A "pair" of alternatives (x, y) means an ordered pair of distinct alternatives $x, y \in A$.

Orderings will sometimes be written in a compact form such as (abc) or $(a\{bc\})$, where (abc) means that a is preferred to b, which is preferred to c. When alternatives appear within braces, no specification of preferences between these alternatives is made. Thus $(a\{bc\})$ means that a is preferred to b and c, and either b is preferred to c or c is preferred to b. We sometimes write $(abc) \in \Omega_i$ or $(ab) \in \Omega_i$ when what we mean more precisely is that $aPbPc$, or aPb, for some $P \in \Omega_i$. The expression $(a\{bc\}) \in \Omega_i$ for all $i \in C$ means that for every $i \in C$ at least one of the orderings (abc), (acb) belongs to Ω_i.

An *Arrow social welfare function* is a function $f: \Pi\Omega_i \to U$ that satisfies the Pareto condition and independence of irrelevant alternatives. The expression $xf(P)y$ means that x is socially strictly preferred to y under profile P. A *voting procedure* is a function $F: \Pi\Omega^n \times \mathscr{P}(A) \to A$, where $\mathscr{P}(A)$ is the power set of A. A voting procedure F satisfies *Property* α iff for all $B \subset C \subset A$, if $F(P, C) = x \in B$, then $F(P, B) = x$. We assume in this paper that all voting procedures satisfy Property α. Arrow SWFs and voting procedures are closely related; indeed, given a member of either class we can construct a member of the other. Let f be an Arrow SWF; we define the *derived voting procedure* F_f by letting $F_f(P, B)$ equal the best element in $B \subset A$ according to the social preference relation $f(P)$. Similarly, we might begin with a voting procedure F. Following Schmeidler and Sonnenschein [13], we define the *derived SWF* f_F pair by pair: for all $x, y \in A$ and $P \in \Pi\Omega_i$, $xf_F(P)y$ iff $x = F(P, \{x, y\})$.

A SWF satisfies the *Pareto condition* iff xP_iy for all i implies $xf(P)y$ for all x, y.

A voting procedure satisfies the *Pareto condition* iff xP_iy for all $y \in B$ for all i implies $F(P; B) = x$. We assume in this paper that all voting procedures satisfy the Pareto condition.

A SWF satisfies *independence of irrelevant alternatives* iff $xP_i y \Leftrightarrow xP_i' y$ for all i implies $xf(P) y \Leftrightarrow xf(P') y$.

A *SWF dictator* is an individual i such that for all x, y, and all $P \in \Pi\Omega_i$, $xP_i y \Rightarrow xf(P) y$.

A *voting procedure dictator* is an individual i such that for all $P \in \Pi\Omega_i$ and all agendas $B \subset A$, $F(P, B) P_i x$ for all $x \in B \backslash F(P, B)$.

A voting procedure is *manipulable* iff there exist a profile $(P_1,..., P_n) \in \Pi\Omega_i$, an ordering $Q_i \in \Omega_i$, and an agenda $B \subset A$ such that $F(P_1,..., P_{i-1}, Q_i, P_{i+1},..., P_n; B) P_i F(P_1,..., P_n; B)$.

A voting procedure is *group manipulable* iff there exist a coalition C and a pair of profiles $(P_1,..., P_n) \in \Pi\Omega_i$ and $(Q_1,..., Q_n) \in \Pi\Omega_i$ such that $P_i = Q_i$ for all $i \notin C$ and $F(Q_1,..., Q_n; B) P_i F(P_1,..., P_n; B)$ for all $i \in C$, for some $B \subset A$.

A SWF is *essential* iff for all i there exist a profile $(P_1,..., P_n) \in \Pi\Omega_i$, an ordering $Q_i \in \Omega_i$, and a pair of alternatives (x, y) such that $xf(P_1,..., P_n) y$ and $yf(P_1,..., P_{i-1}, Q_i, P_{i+1},..., P_n) x$.

A voting procedure is *essential* iff for all i there exist a profile $(P_1,..., P_n) \in \Pi\Omega_i$, an ordering $Q_i \in \Omega_i$, and an agenda $B \subset A$ such that $F(P_1,..., P_n; B) \neq F(P_1,..., P_{i-1}, Q_i, P_{i+1},..., P_n; B)$.

A SWF is *monotonic* iff for all x, y and for all coalitions C, if $xP_i y$ for all $i \in C$ implies $xf(P) y$, then $xP_i' y$ for all $i \in D \supset C$ implies $xf(P') y$.

3. Domains Permitting Construction of Essential Procedures

In this section we give a characterization of the domains of admissible preferences that permit the construction of well-behaved essential procedures. Before doing so, we show that our conditions for essential procedures are more restrictive than the Kalai–Muller conditions for nondictatorial rules, at least in their context of common admissible preference sets for all individuals. In general our condition is not comparable to the Kalai–Muller property, since we do not require identical admissible orderings for all individuals.

Let $A = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, and suppose $n \geqslant 3$. Consider the domain Ω^n consisting of all n-tuples of orderings of the form $(\{x_1 x_2 x_3\}\{y_1 y_2 y_3\})$. That is, the x's always appear above the y's in permissible orderings; the sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are free triples, so all logically possible orderings of members of these sets are permissible. Any SWF satisfying the weak Pareto condition ranks x_i above y_j for $i, j = 1, 2, 3$. By a theorem of Murakami [11], any Arrow SWF on this domain must have a dictator on each of the free triples. If we allow different individuals to dictate on the x's and the y's, then we have constructed a rule that has no dictator in Arrow's sense. But all other individuals (of whom there is at least one) must be powerless under any such SWF. Hence no essential SWF exists on this domain when there are more than two individuals. Clearly we can adapt this domain to make a similar point for arbitrary numbers of individuals by stacking additional free triples below $\{y_1, y_2, y_3\}$. Obviously a nondictatorial SWF exists on any restricted domain when an essential SWF exists, since essentiality is a stronger condition.

The set of ordered pairs of distinct alternatives is *essentially decomposable* (ED) *with respect to a domain* $\Pi\Omega_i$ if and only if for all such pairs (x, y) there exists a nonempty set $W(x, y)$ of coalitions $C \subset N$ such that

(1) For all coalitions C, if $(xy) \in \Omega_i$ for all $i \in C$ and $(yx) \in \Omega_i$ for all $i \in N \backslash C$, then $C \in W(x, y)$ if and only if $N \backslash C \notin W(y, x)$.

(2) For all coalitions C and B such that $C \subset B$, $C \in W(x, y)$ implies $B \in W(x, y)$.

(3) For all $i \in N$, there exist a pair (x, y) and a coalition C such that (xy), $(yx) \in \Omega_i$, $C \in W(x, y)$, and $C \backslash \{i\} \notin W(x, y)$.

(4) For all x, y, z, if $C \in W(x, y)$, $B \in W(y, z)$, $D \equiv \{i \in C: (zxy) \notin \Omega_i\}$, $E \equiv \{i \in B: (yzx) \notin \Omega_i\}$, $(xy) \in \Omega_i$ for all $i \in C$, $(yz) \in \Omega_i$ for all $i \in B$, and $(zx) \in \Omega_i$ for all $i \in N \backslash C \backslash B$, then $\varnothing \neq D \cup E \cup (C \cap B) \in W(x, z)$.

Our principal result can now be stated.

THEOREM 1. *The following four conditions are equivalent*:

(a) *The set of pairs is essentially decomposable with respect to the domain.*

(b) *The domain permits construction of an n-person essential monotonic Arrow social welfare function.*

(c) *The domain permits construction of an n-person essential individually-nonmanipulable voting procedure.*

(d) *The domain permits construction of an n-person essential group-nonmanipulable voting procedure.*

Proof. We establish the theorem by proving the equivalence of (a) to (b), then of (b) to (c), and finally of (c) to (d). We begin with a lemma showing that essentiality is preserved by taking the derived rule.

LEMMA 1. *If f is an essential Arrow SWF, then the derived voting procedure F_f is essential. If F is an essential voting procedure, then the derived Arrow SWF f_F is essential.*

Proof. The proof is a straightforward application of the definitions and is omitted.

Proof that (a) $\Leftrightarrow$ (b): Suppose that the set of pairs is essentially decomposable with respect to the domain $\Pi\Omega_i$. We construct an n-person essential, monotonic Arrow SWF in the following way:

$$xf(P)\,y \Leftrightarrow \{i \in N: xP_i\,y\} \in W(x, y).$$

That is, for all pairs of alternatives (x, y) we construct the rule by defining the coalitions belonging to $W(x, y)$ as the winning coalitions for that pair.

Condition (1) in the definition of decomposability implies that $f(P)$ is an asymmetric and complete binary relation. Since f is defined pair by pair, it satisfies independence of irrelevant alternatives. By condition (2), f is monotonic. The function f is essential since essentiality is equivalent to Condition (3). Since $W(x, y)$ is a nonempty and monotonic set of coalitions, $N \in W(x, y)$; therefore f satisfies the Pareto condition.

Finally we need to show that $f(P)$ is transitive. Suppose that $xf(P)\,yf(P)z$; we must prove that $xf(P)z$. Denote by C the set of individuals i such that $xP_i\,y$ in profile P, and let B equal the set of individuals i such that yP_iz in profile P. Clearly $C \in W(x, y)$ and $B \in W(y, z)$. By the definitions of C and B, $(xy) \in \Omega_i$ for all $i \in C$ and $(yz) \in \Omega_i$ for all $i \in B$. All members of the (possibly empty) set $N\backslash C\backslash B$ prefer y to x and z to y; thus $(zx) \in \Omega_i$ for all $i \in N\backslash C\backslash B$. By Condition (4), $\varnothing \neq D \cup E \cup (C \cap B) \in W(x, z)$. Each member of $D \subset C$ prefers x to y, but (zxy) is not permissible ordering for D. Hence the preferences of each member of D must be describable as $(x\{yz\})$. Similarly, each member of $E \subset B$ prefers y to z, but (yzx) is not a permissible ordering for D. Thus E's preferences must be describable as $(\{xy\}z)$. Therefore all members of $D \cup E$ prefer x to z. So do the members of $C \cap B$, since each prefers x to y and y to z. Since x is preferred to z by all members of $D \cup E \cup (C \cap B) \in W(x, z)$, we conclude that $xf(P)z$.

This establishes that (a) $\Rightarrow$ (b). Suppose now that on a domain $\Pi\Omega_i$ there exists an n-person essential monotonic Arrow SWF. We want to show that the set of pairs is decomposable with respect to $\Pi\Omega_i$.

For each pair (x, y), define $W(x, y)$ as the set of all coalitions that are decisive over that pair, that is, the set of coalitions whose members, by unanimously preferring x to y, can ensure a social ranking of x over y regardless of the preferences of others.

The Pareto condition requires that $N \in W(x, y)$ for all (x, y). Thus $W(x, y)$ is nonempty for all pairs.

To show that Condition (1) holds, consider the following profile:

$$C\colon (xy), \qquad N\backslash C\colon (yx).$$

If $C \in W(x, y)$ and $N\backslash C \in W(y, x)$, then the (strict) social preference relation is not asymmetric. If $C \notin W(x, y)$ and $N\backslash C \notin W(y, x)$, then the social preference relation is incomplete at this profile. Thus Condition (1) holds.

Consider Condition (2); suppose $C \in W(x,y)$ and $C \subset B$. If for some $i \in B$ $(xy) \notin \Omega_i$, then vacuously $B \in W(x,y)$. If not, then $B \in W(x,y)$ by monotonicity.

The essentiality and monotonicity of the SWF require that each individual be a pivotal member of some minimal winning coalition for some pair of alternatives. This is exactly what Condition (3) requires. It remains only to argue that Condition (4) holds.

Suppose then that $C \in W(x,y)$, $B \in W(y,z)$, $(xy) \in \Omega_i$ for all $i \in C$, $(yz) \in \Omega_i$ for all $i \in B$, and $(zx) \in \Omega_i$ for all $i \in N \backslash C \backslash B$. We have to show that $\varnothing \neq D \cup E \cup (C \cap B) \in W(x,z)$.

We first show that $D \cup E \cup (C \cap B) \neq \varnothing$. Suppose not. Then $(zxy) \in \Omega_i$ for all $i \in C$, since $D = \varnothing$, and $(yzx) \in \Omega_i$ for all $i \in B$, since $E = \varnothing$. Consider the following permissible profile P:

$$C\colon (zxy), \qquad B\colon (yzx), \qquad N \backslash C \backslash B\colon (zx).$$

Since xP_iy for all $i \in C$ and $C \in W(x,y)$, we can conclude that $xf(P)y$. Similarly, since yP_iz for all $i \in B$ and $B \in W(y,z)$, it must be true that $yf(P)z$. By transitivity, $xf(P)z$. But this contradicts the Pareto condition, since zP_ix for all $i \in N$. Thus $D \cup E \cup (C \cap B) \neq \varnothing$.

Consider now the following P:[2]

$$\begin{array}{rl} D\colon (x\{yz\}), & E\colon (\{xy\}z), \\ C \backslash B \backslash D\colon (zxy), & B \backslash C \backslash E\colon (yzx), \\ C \cap B\colon (xyz), & N \backslash C \backslash B\colon (zx). \end{array}$$

Each of these preferences is permissible for the group in question by assumption and the definitions of D and E. Since xP_iy for all $i \in C$ and $C \in W(x,y)$, it follows that $xf(P)y$. Since yP_iz for all $i \in B$ and $B \in W(y,z)$, we conclude that $yf(P)z$. By transitivity, $xf(P)z$. But only the coalition $D \cup E \cup (C \cap B)$ prefers x to z in this profile. By monotonicity and independence, whenever xP_iz for all $i \in D \cup E \cup (C \cap B)$ it must be that x is socially preferred to z. Thus $D \cup E \cup (C \cap B) \in W(x,z)$; hence Condition (4) holds.

Proof that (b) $\Leftrightarrow$ (c). Suppose there exists an essential nonmanipulable voting procedure F on a domain $\Pi\Omega_i$. We will show the existence of a suitable SWF by noting that the derived SWF f_F satisfies the conditions advertised in (b). Lemma 1 guarantees the essentiality of f_F. Schmeidler and

[2] Clearly no difficulty arises if either $C \cap B \cap D$ or $C \cap B \cap E$ is nonempty, since the preference (xyz) is consistent with the descriptions $(x\{yz\})$ and $(\{xy\}z)$.

Sonnenschein [13] (among others) have shown in the unrestricted domain case that nonmanipulability of the voting procedure implies the satisfaction by the derived SWF of independence of irrelevant alternatives. This argument remains true when the domain is restricted: the SWF satisfies independence on $\Pi\Omega_i$ when the voting procedure is nonmanipulable on that domain. Transitivity follows from single-valuedness and Property α; f satisfies the Pareto condition since F does.

To prove the converse, we will show that if a SWF f satisfies transitivity, Pareto independence, and essentiality on $\Pi\Omega_i$, then the voting procedure F_f derived from it is nonmanipulable and essential. The essentiality of F_f is guaranteed by Lemma 1. The transitivity of f implies that F_f satisfies Property α. Suppose that F_f is manipulable by voter i. Then there exists a pair of alternatives (x, y) and two profiles $P = (P_1, ..., P_n)$ and $Q = (P_1, ..., P_{i-1}, Q_i, P_{i+1}, ..., P_n)$ such that $x = F_f(P, \{x, y\})$, $y = F_f(Q, \{x, y\})$, and yP_ix. Thus $xf(P)y$ and $yf(Q)x$. If yQ_ix, then f violates independence, since the restriction of P to $\{x, y\}$ equals the restriction of Q to $\{x, y\}$. If xQ_iy, then f is nonmonotonic, since when voter i changes his reported preference from yP_ix to xQ_iy, the social ordering changes from $xf(P)y$ to $yf(Q)x$. In either case, our assumptions about f have been contradicted.

Proof that (c) $\Leftrightarrow$ (d). Since groups may have single members, (d) implies (c) directly. Suppose, then, that an n-person essential voting procedure on the domain is manipulable by a group. That is, there exists a coalition $C = \{1, ..., k\}$ (with suitable renumbering), a set of alternatives B, and two admissible profiles $P = (P_1, ..., P_n)$ and $(Q_1, ..., Q_k, P_{k+1}, ..., P_n)$ such that $F(P; B) = x$, $F(Q_1, ..., Q_k, P_{k+1}, ..., P_n; B) = y$, and yP_ix for all $i \in C$. By Property α, $F(P; \{x, y\}) = x$ and $F(Q_i, ..., Q_k, P_{k+1}, ..., P_n; \{x, y\}) = y$. Let $z_i = F(Q_1, ..., Q_i, P_{i+1}, ..., P_n; \{x, y\})$, $i = 1, ..., k$; let j be the smallest i such that $z_i = y$. Then F is manipulable by individual j at the profile $(Q_1, ..., Q_{j-1}, P_j, ..., P_n)$. ∎

4. Other Decomposability Conditions

In this section we relate our work to two other decomposability conditions that have appeared in the literature. One of them was defined by Kalai and Ritz [5], who study what they call "private alternatives domains," which consist of profiles of preferences over n-dimensional vectors. We define a variant of the ED condition that permits us to generalize the Kalai–Ritz nondictatorial decomposability property. The required changes in ED revolve around the necessity of introducing a restricted form of individual indifference as a result of the vector structure of the alternatives space.

The other decomposability condition was defined by Kalai and Muller [4] for nondictatorial procedures; it has been discussed extensively in Section 1.

We show here that when the sets of admissible preferences are identical, that is, $\Omega_i = \Omega$ for all i, and when the number of individuals is two, then our ED condition is equivalent to the nondictatorial decomposability condition. This proof, together with Theorem 1 and our demonstration of the equivalence of nondictatorship and two-person essentiality in the common-domain case, provides an independent proof of the Kalai–Muller result.

In the Kalai–Ritz model, each individual i has a strong preference ordering $>_i$ over a set of private alternatives X. The set of social alternatives A is a subset of X^n. Individual i is concerned only with the ith component of the social alternative $x = (x_1, ..., x_n)$. His preferences P_i over A are induced by his preferences over X as follows: xI_iy (that is, i is indifferent between x and y) if and only if $x_i = y_i$; xP_iy if and only if $x_i >_i y_i$. This formulation introduces only a limited type of indifference; i is indifferent between x and y in some permissible ordering if and only if he is indifferent between them in all permissible orderings, since it must be true that $x_i = y_i$. Consideration of the usual form of indifference would be much more complex.

We shall call a SWF defined on profiles of preference orderings constructed in this way a *multidimensional social welfare function* (MSWF); such functions are equivalent to the aggregation procedures considered by Kalai and Ritz. Let $I(x, y) = \{i \in N: x_i = y_i\}$, where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Thus $I(x, y)$ is the set of individuals who are indifferent between x and y under any profile.

An MSWF satisfies the *strong Pareto condition* if xP_iy for all $i \in N \backslash I(x, y)$ implies $xf(P)y$ for all x, y.

Definitions of other SWF properties carry over to MSWFs without alteration. We require that any MSWF satisfy the strong Pareto condition and independence of irrelevant alternatives.

The set of ordered pairs of distinct n-dimensional alternatives satisfies the *essential decomposability condition for MSWFs* (MED) *with respect to a domain* $\Pi\Omega_i$ if, for all such pairs (x, y), there exists a nonempty set $W(x, y)$ of coalitions $C \subset N$ such that

(1) For all coalitions C and all pairs (x, y), if $(xy) \in \Omega_i$ for all $i \in C \backslash I(x, y)$ and $(yx) \in \Omega_i$ for all $i \in N \backslash C \backslash I(x, y)$, then $C \backslash I(x, y) \in W(x, y)$ if and only if $N \backslash C \backslash I(x, y) \notin W(y, x)$.

(2) For all coalitions C and B such that $C \subset B$, $C \in W(x, y)$ implies $B \in W(x, y)$.

(3) For all $i \in N$ there exist a pair (x, y) and a coalition C such that $(xy), (yx) \in \Omega_i$, $C \in W(x, y)$, and $C \backslash \{i\} \notin W(x, y)$.

(4) For all x, y, z, suppose $C \in W(x, y)$ and $B \in W(y, z)$. Let $D = \{i \in C: (zxy) \notin \Omega_i\}$ and $E = \{i \in B: (yzx) \notin \Omega_i\}$. If $(xy) \in \Omega_i$ for all $i \in C$, $(yz) \in \Omega_i$ for all $i \in B$, and $(zx) \in \Omega_i$ for all $i \in N \backslash C \backslash B \backslash I(x, z)$, then $\varnothing \neq [D \cup E \cup (C \cap B)] \backslash I(x, z) \in W(x, z)$.

The following proposition generalizes the Kalai–Ritz result in two ways: first, our condition guarantees the existence of an essential (rather than a nondictatorial) MSWF and, second, our result allows for differences in individuals' admissible preferences over the private alternatives X.

THEOREM 2. *The set of pairs is MED with respect to a domain if and only if the domain permits construction of an n-person essential monotonic MSWF.*

Proof. Suppose the set of pairs is MED. We construct an n-person essential MSWF as follows

$$xf(P)\,y \Leftrightarrow \{i \in N: xP_i y\} \in W(x, y).$$

We need only to demonstrate the transitivity of f since the rest of the proof is essentially the same as the proof of Theorem 1. Suppose that $xf(P)\,yf(P)\,z$. Let $C = \{i \in N: xP_i y\}$, $B = \{i \in N: yP_i z\}$. Clearly $C \in W(x, y)$, $B \in W(y, z)$, $(xy) \in \Omega_i$ for all $i \in C$, and $(yz) \in \Omega_i$ for all $i \in B$. All members of the set $N \backslash C \backslash B \backslash I(x, y)$ prefer y to x and z to y. Thus for all such persons i, $(zx) \in \Omega_i$.

A member of D prefers x to y but $(zxy) \notin \Omega_i$ for all $i \in D$. Thus he either belongs to $I(x, z)$ or his preference may be described as $(x\{yz\})$. Similarly a member of E is either a member of $I(x, z)$ or his preference is describable as $(\{xy\}z)$. Since the members of $C \cap B$ prefer x to y to z, all members of $D \cup E \cup (C \cap B) \backslash I(x, z)$ prefer x to z. Since this coalition, by Condition (4), is nonempty and belongs to $W(x, z)$, we conclude that $xf(P)z$.

As for the converse, suppose the domain permits construction of an n-person essential monotonic MSWF. We wish to show that the domain is MED. Conditions (1)–(3) can be shown to hold as in Theorem 1.

Suppose the assumptions of Condition (4) are satisfied. Suppose also that $[D \cup E \cup (C \cap B)] \backslash I(x, z) = \varnothing$; we shall show that this leads to a contradiction. Since $(xy) \in \Omega_i$ for all $i \in C$ and $(yz) \in \Omega_i$ for all $i \in B$, $(xyz) \in \Omega_i$ for all $i \in C \cap B$. Therefore $(C \cap B) \cap I(x, z) = \varnothing$. Hence

$$\begin{aligned}
\varnothing &= [D \cup E \cup (C \cap B)] \backslash I(x, z) \\
&= [D \backslash I(x, z)] \cup [E \backslash I(x, z)] \cup [(C \cap B) \backslash I(x, z)] \\
&= [D \backslash I(x, z)] \cup [E \backslash I(x, z)] \cup (C \cap B).
\end{aligned}$$

Thus each of the sets $D \backslash I(x, z)$, $E \backslash I(x, z)$, and $C \cap B$ is empty, so $D \subset I(x, z)$ and $E \subset I(x, z)$. By the definition of D, $(zxy) \in \Omega_i$ for all $i \in C \backslash D$. It follows that $(C \backslash D) \cap I(x, z) = \varnothing$. This implies that $C \cap [D \cup I(x, z)] = D$. Since $D \subset I(x, z)$, we know then that $C \cap I(x, z) = D$. Similarly $B \cap I(x, z) = E$.

Consider, then, the following permissible profile P:

$$D\colon (xy) \text{ and } xI_i z, \qquad C\backslash D\colon (zxy),$$
$$E\colon (yz) \text{ and } xI_i z, \qquad B\backslash E\colon (yzx),$$
$$N\backslash C\backslash B\backslash I(x,z)\colon (zx).$$

By the strong Pareto condition $zf(P)x$. But $xf(P)y$ since $C \in W(x, y)$ and $yf(P)z$ since $B \in W(y, z)$. This contradicts the transitivity of f, proving that $[D \cup E \cup (C \cap B)]\backslash I(x, z) \neq \varnothing$.

We now must show that $[D \cup E \cup (C \cap B)]\backslash I(x, z) \in W(x, z)$. Consider the following profile P:

$$D\backslash I(x,z)\colon (x\{yz\}), \qquad D \cap I(x,z)\colon (xy) \text{ and } xI_i z,$$
$$E\backslash I(x,z)\colon (\{xy\}z), \qquad E \cap I(x,z)\colon (yz) \text{ and } xI_i z,$$
$$C\backslash B\backslash D\colon (zxy), \qquad B\backslash C\backslash E\colon (yzx),$$
$$C \cap B\colon (xyz), \qquad N\backslash C\backslash B\backslash I(x,z)\colon (zx).$$

All members of $C \in W(x, y)$ prefer x to y; all members of $B \in W(y, z)$ prefer y to z. Thus $xf(P)yf(P)z$; by transitivity, $xf(P)z$. But only the members of $[D \cup E \cup (C \cap B)]\backslash I(x, z)$ prefer x to z in this profile; hence that set belongs to $W(x, z)$. ∎

Kalai and Ritz require that each individual have the same set of admissible preferences $>_i$ over the vector components X. It is worth emphasizing, however, that the sets Ω_i of admissible induced preferences P_i on the vectors A will generally differ. Consider the two alternatives $x = (x_1, x_2)$ and $y = (x_1, y_2)$, with $x_2 \neq y_2$. The set Ω_2 may well include (xy) and (yx), depending on the admissible preferences over components, but it must be identically true that $xI_1 y$, even though individual 1 has the same admissible preferences over components as individual 2. In Theorem 2, in contrast with Kalai and Ritz, we do not impose the restriction that admissible component preferences be identical.

In the remainder of this section we turn to the Kalai–Muller decomposability condition for the existence of nondictatorial SWFs on restricted domains. We prove two lemmas, each applying to the case when the number of individuals is two and when the admissible preference sets are identical for the individuals, that is $\Omega_1 = \Omega_2 = \Omega$. In those circumstances, we show first that the ED condition is equivalent to the Kalai–Muller nondictatorial decomposability property. Next we prove that under those conditions a SWF satisfying the Pareto condition is nondictatorial if and only if it is essential.

These lemmas serve two purposes. First, they clarify the relationship between these two seemingly different domain restrictions. Second, together with Theorem 1, they stand as an alternative proof of the Kalai–Muller theorem.

In this common-admissible-preferences context a pair (x, y) is *nontrivial* if there exist $P, Q \in \Omega$ such that xPy and yQx. A set of pairs is *nontrivial* if it contains at least one nontrivial pair. With these definitions in mind we can now state Kalai and Muller's decomposability condition; this version is the form described in a remark in [4, p. 465].

The set of pairs is *nondictatorially decomposable* (ND) *with respect to a domain* Ω^n if there exist two nontrivial sets R_1, $R_2 \subset A^2$ such that

(a) For all nontrivial pairs (x, y), $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$.

(b) For all nontrivial pairs the following are true for $i = 1, 2$:

(b1) If (xyz), $(yzx) \in \Omega$, then $(x, y) \in R_i$ implies $(x, z) \in R_i$.

(b2) If (xyz), $(yzx) \in \Omega$, then $(z, x) \in R_i$ implies $(y, x) \in R_i$.

(b3) If $(xyz) \in \Omega$, then $(x, y) \in R_i$ and $(y, z) \in R_i$ imply $(x, z) \in R_i$.

LEMMA 2. *Suppose $n = 2$ and $\Omega_1 = \Omega_2 = \Omega$. Then the set of pairs is ED with respect to Ω^2 if and only if it is ND with respect to Ω^2.*

Proof. Suppose ED holds. Define $R_i = \{(xy) \in A^2 : \{i\} \in W(x, y)\}$, for $i = 1, 2$. Condition (a) follows directly from ED Condition (1). Condition (3) of ED implies that each R_i contains at least one nontrivial pair, so each R_i is nontrivial.

To show that Condition (b1) holds, suppose (xyz), $(yzx) \in \Omega$. Let $C = \{i\} \in W(x, y)$; we know that $B = \{1, 2\} \in W(y, z)$. Since $E = \varnothing$ and $D \subset \{i\} = C$, Condition (4) implies that $\{i\} \in W(x, z)$, as required.

Condition (b2) follows by a similar argument, replacing x with y, y with z, and z with x. Condition (b3) follows immediately from Condition (4) if we let $B = C = \{i\}$.

Suppose now ND holds. Define $W(x, y)$ for all pairs (x, y) to include precisely two coalitions: $\{1, 2\}$ and $\{i\}$ such that $(x, y) \in R_i$.

Condition (1) is implied by Condition (a). Condition (2) is readily satisfied since any two-person SWF is monotonic if it satisfies the Pareto condition. Condition (3) follows from the fact that each R_i is nontrivial.

As for Condition (4), first let $C = \{i\}$ and $B = \{1, 2\}$. Then $E = \varnothing$ and $D \subset \{i\}$. Thus $D \cup E \cup (C \cap B) = \{i\}$; by Condition (b1), $\{i\} \in W(x, z)$.

If $C = \{1, 2\}$ and $B = \{i\}$, a similar argument (replacing x with z, y with x, and z with y) using Condition (b2) shows that $D \cup E \cup (C \cap B) = \{i\} \in W(x, z)$.

Finally, if $C = B$, Condition (b3) implies that $D \cup E \cup (C \cap B) = C = B \in W(x, z)$. ∎

LEMMA 3. *Suppose* $n=2$ *and* $\Omega_1=\Omega_2=\Omega$. *Then an aggregation procedure satisfying the Pareto condition is nondictatorial if and only if it is essential.*

Proof. We prove the lemma only for SWFs; the argument for voting procedures is quite similar. If a SWF is essential, then for individual 1 there exists a nontrivial pair (x, y), a profile $(P_1, P_2) \in \Omega^2$, and an ordering $Q \in \Omega$ such that $xf(P_1, P_2)\,y$ and $yf(Q, P_2)\,x$. Thus at one of these profiles the social ranking of (x, y) differs from the preferences of individual 2. Hence individual 2 is not a dictator; neither is individual 1 by an identical argument.

Suppose a SWF is inessential; for definiteness suppose individual 2 never affects the outcome. For any $P \in \Omega$, $f(P, P) = P$, by the Pareto condition. But since 2 is inessential, $f(P, Q) = P$ for all $Q \in \Omega$. Thus for all permissible profiles the social preference relation coincides with individual 1's ordering; hence 1 is a dictator. ∎

It is worth pointing out that nondictatorship does not imply essentiality when individuals have different sets of admissible preferences. For example, let $\Omega_1 = \{(xyz)\}$, $\Omega_2 = \{(zyx)\}$, and $f(P) = (yzx)$ for the unique admissible profile. Neither individual can affect the social preference relation, yet the social ordering coincides with neither individual's fixed preferences. The Pareto condition is of course vacuous for this domain, since the two individuals can never agree on any pair of alternatives. This example highlights the weakness of nondictatorship as a guarantor of individual power in a restricted domain context.

It is clear that Lemmas 2 and 3 and Theorem 1 provide an alternative proof of the following result:

PROPOSITION (Kalai and Muller). *The set of pairs is nondictatorially decomposable with respect to a domain if and only if the domain permits construction of a two-person monotonic Arrow social welfare function.*

Kalai and Muller show in addition, as we have noted, that the existence of a monotonic Arrow social welfare function is independent of the number of individuals.

5. NEUTRAL ESSENTIAL PROCEDURES

We examine in this section the consequences of requiring aggregation procedures to be neutral, that is, to treat alternatives symmetrically. A *decisive set for* (x, y) under an aggregation procedure is a coalition that, when each of its members prefers x to y in a profile P, can ensure that $xf(P)y$ (in the case of SWFs) or $x = F(P, \{x, y\})$ (in the case of voting procedures). An aggregation procedure is *neutral* if the family of decisive sets for (x, y) is the same for all pairs (x, y). (On an unrestricted domain every Arrow SWF is neutral in this sense, but as we have seen nonneutral ones exist on restricted domains.) A more conventional definition of neutrality for SWFs is that any permutation of the names of the alternatives applied to each individual's ordering results in an identical permutation of the names of the alternatives in the social ordering. For SWFs satisfying independence of irrelevant alternatives, the two definitions are equivalent when the domain is unrestricted. We have adopted the former definition for the restricted domain case in part because a permuted admissible profile may itself be inadmissible.

We give a necessary and sufficient condition for a domain with common admissible preferences for each individual to be sufficiently sparse to permit the construction of neutral essential aggregation procedures. Before doing so, we note that this strengthening of ED is not vacuous. At the beginning of Section 3 we gave an example of a domain on which exist essential SWFs but not neutral ones. To satisfy essentiality the triple-dictators in that example must be different on each triple, contradicting neutrality.

Neutral–Essential Decomposability (NED): For no $x, y, z \in A$ are each of (xyz), (yzx), and (zxy) admissible.

THEOREM 3. *The following conditions are equivalent*:

(a) *The domain satisfies NED.*

(b) *The domain permits construction of an n-person neutral essential monotonic Arrow social welfare function.*

(c) *The domain permits construction of an n-person neutral essential nonmanipulable voting procedure.*

Proof. We show only the equivalence of (a) and (b). The equivalence of (b) and (c) can be proved easily by methods similar to those used in the proof of Theorem 1.

(a) $\Rightarrow$ (b): Majority rule (with an individual who gets an extra vote to break ties when needed if there is an even number of voters) satisfies the three stated properties of a SWF. The property of NED implies value restriction holds for all admissible profiles; this guarantees the transitivity of strict majority preference.

(b) $\Rightarrow$ (a): Suppose NED fails to hold. Let D be a decisive set of minimal cardinality (which must exist because of the Pareto condition and the finiteness of the voter set). Then D has at least two members (or else neutrality would imply the existence of a dictator). Choose $i \in D$ and consider the following profile P:

$$\{i\}: (xyz), \qquad D\backslash\{i\}: (yzx), \qquad N\backslash D: (zxy).$$

Then $xf(P)y$ is the social outcome between x and y, since otherwise $D\backslash\{i\}$ is decisive, contradicting D's minimality. Also $yf(P)z$ follows from D's decisiveness, which by neutrality applies to all pairs. But $zf(P)x$ is the social outcome since otherwise $\{i\}$ is decisive. This contradicts transitivity. ∎

We remark that NED is almost a sufficient condition for the constructibility of an anonymous SWF, since majority rule (without a tie-breaker) satisfies that condition. When the electorate is odd, NED is sufficient, since ties cannot occur and thus need not be broken.

Our proof makes clear as well that NED is also necessary and sufficient for the existence of a nondictatorial (rather than essential) neutral SWF on a domain. The necessity part of the theorem, it should also be pointed out, bears a close relationship to Sen's Theorem 10*1 [14, p. 177].

6. Concluding Remarks

Two further points about this branch of the domain restriction literature require discussion. The first is a modeling advantage of the decomposability approach in applications of domain-restriction results to more concrete economic and political situations. Our conditions, as well as those of Kalai, Muller, and Ritz, are restrictions on permissible preferences for *individuals*. The Sen–Pattanaik and related preference restrictions are conditions on *combinations* of individual orderings. To argue in some context that, say, Sen and Pattanaik's value restriction is satisfied requires specification of the social processes that always lead to the profile restrictions. (An important exception is the case of concave preferences over a unidimensional choice space.) Our restrictions, which are not prohibitions on combinations, require only modeling of individual behavior.

The second point concerns the important question of the likelihood in some sense that these conditions can be satisfied. That is, are our theorems and those of Kalai–Muller and Kalai–Ritz *possibility* theorems or *impossibility* theorems? We offer some partial answers here for the case in which all individuals have the same set of admissible preferences, that is, $\Omega_i = \Omega$ for all i.

The close relationship between NED and majority rule suggests that this domain restriction will be quite difficult to satisfy. The transitivity of majority rule is a fragile property in applied politico-economic models. Yet as we showed in Section 5, NED implies this stringent condition when the number of persons is odd. Our best hope for finding a useful domain restriction thus seems to lie in the abandonment of neutrality. In any event, neutrality is not a particularly compelling ethical requirement to impose on aggregation procedures; as Arrow [1, p. 101] argues, "neutrality is not intuitively basic." We turn then to the difficulty of satisfying essential decomposability.

One way to pose the question is to ask how large Ω can be if we require the set of pairs to be essentially decomposable with respect to Ω^n. Suppose there are m alternatives. Kim and Roush [7, 8] have shown that if nondictatorial decomposability holds for Ω^n, then Ω can contain no more than $m!/2 + (m-1)!$ orderings. Moreover Kim and Roush have shown that an Ω satisfying ND described by Kalai and Ritz [6] is of exactly this size. This set of orderings is easy to describe. An *inseparable pair* is a nontrivial ordered pair (x, y) such that $(xzy) \in \Omega$ for no $z \in A$. Kalai and Ritz show that any Ω^n such that Ω contains an inseparable pair satisfies ND. One such Ω attains the Kim–Roush bound number. The nondictatorial SWFs Kalai and Ritz describe for such domains make some individual i a dictator on all pairs except (x, y) and use an arbitrary decision rule for (x, y) that is different from dictatorship by individual i.

Suppose, however, that we choose a particular decision rule for (x, y): let $xf(P)y$ if and only if xP_iy for all $i \in N$. This SWF is not merely nondictatorial; it is essential as well, since every individual is a pivotal member of $N \in W(x, y)$. It follows that the Kalai–Ritz inseparable-pair condition implies essential decomposability, and that $m!/2 + (m-1)!$ is the size of the largest Ω compatible with ED.

Since there are $m!$ logically possible orderings of m alternatives, we conclude *there exist essential domains in which more than half of all possible orderings are admissible.* This conclusion is a striking positive result on the attainability of essential decomposability.

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