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Capital Accumulation Games of Infinite Duration

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Consider a market in which firms accumulate capital according to the Nerlove–Arrow capital accumulation equation. Each player chooses a path of investment and thus an induced path of capital to maximize his total discounted profits which depend on his own capital and the capital stocks of his rivals. Existence is proved for such a nonzero sum, infinite horizon differential game and conditions under which the game converges to a particular stationary point, regardless of the initial conditions are shown. Thus, the game possesses the property of conditional global asymptotic stability. *Journal of Economic Literature*, Classification Numbers: 022, 611.

1. INTRODUCTION

The main purpose of this paper is to investigate a class of games in which each player accumulates some form of capital. The payoff of each player depends on his own capital and the capital stocks of his rivals. Changes in stock, however, are not instantaneous. The firm can invest in the capital stock and it deteriorates at a certain constant proportional rate. Each player thus chooses a path of investment and thus an induced path of capital accumulation so as to maximize his total discounted profits.

The first issue in such a game is the problem of existence of paths which form a Nash solution, i.e., given the paths of the rivals, the firm's strategy is the best response for these paths.

When the existence issue is solved, the main issue is whether such markets have a stationary equilibrium and whether the market will converge to the stationary point. It is straightforward to show that even if a stationary equilibrium path exists, in the finite horizon case the market will not converge to the stationary equilibrium point. Thus the issue of convergence necessitates introducing infinite horizons. To clarify the economic situations of our games, the following are examples that fall into our general class.

EXAMPLE 1 (Durable good production). Consider a market for a durable good. The rental price p is a function of the total stock in the

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market. Firm i can change its stock of durables, $Q_i(t)$, by producing $x(t)$ units at time t and its stock depreciates at a constant rate of δ. Thus its equation is $\dot{Q}_i = x_i - \delta_i Q_i$. Its revenues at time t are given by $p(Q_1 + Q_2)Q_i$ and its cost of production is $C_i(x)$.

EXAMPLE 2 (Advertising and goodwill). Consider a market in which the firms accumulate goodwill G_i according to the Nerlove–Arrow equation $\dot{G}_i = a_i - \delta_i G_i$, were a_i is the advertising investment and δ_i the depreciation due to forgetting and other reasons. Sales of firm i will be some concave function of its relative market share. Price will be determined by a Cournot-type solution. Thus the revenues of firm i is given by $f_i(G_i/\sum_j G_j)$ for some concave function f_i. This subject is dealt with separately by the authors [10].

This work is an extension of two separate lines of research: capital accumulation and differential games.

The capital accumulation equation which is used in this paper was originally investigated by Nerlove and Arrow [20]. Arrow [1, 2] has generalized his original findings by considering two extensions: the first considers a general decay which is not necessarily exponential, and the second considered a nonstationary economic environment. Gould [13], by considering the model of Nerlove–Arrow with strictly convex cost has found that for any initial value of the stock of capital, there exists an initial investment such that the induced capital path converges to a stationary point.

At the same time a whole stream of related research began investigating the stability properties of capital accumulation growth models. In particular, the interest was in finding conditions under which a capital growth system would converge to a particular stationary point regardless of the initial conditions. Such a system was defined as having the global asymptotic stability property. See, for example, the special issue of *J. Economic Theory* (February 1976) and in particular Cass and Shell [6] and Brock and Scheinkman [5]. The common type of condition that relates these works is that more than strict convexity (concavity) is needed.[1]

We are interested in extending the issue posed by Gould. His type of stability can be denoted by *conditional global asymptotic stability* which is weaker than global asymptotic stability since the path converges just for a particular initial condition of investment. In our game global asymptotic stability is ruled out since it can be shown that the game does not even

[1] A function f is more convex than g if $f - g$ is convex. The functions that are needed in these cases are functions which are more convex than quadratic function.

possess local stability. We do, however, investigate the issue of conditional local and global stability.

In terms of differential games, we choose to formulate an open loop solution although it is known to have some limitations; see Spence [24] or Kydland [17]. The closed loop solutions, however, are known to exist only with severe limitations on the structure and duration of the game, for example, Reinganum [21]. The existence issue for zero sum differential games has been extensively investigated. For a review and summary of this line of research, see Friedman [12].

For open loop, nonzero sum, differential games, Scalzo [22] first proved existence for any finite duration. Proofs of existence prior to his work were known only for "small" duration. Scalzo's work has been extended by Wilson [26] and Williams [25] to games with incomplete information and by Scalzo and Williams [23] to games with nonlinear state equations. All three extensions dealt with the finite horizon case.

The issue of conditional local and global asymptotic stability of differential games has been recently investigated in three interesting papers: Brock [4], Flaherty [11], and Haurie and Leitmann [14]. These works assumed the existence of a solution to the particular differential game investigated.[2] Flaherty showed conditions for local stability of a linear–quadratic game. Brock and Haurie and Leitmann studied a family of games that is richer than the one studied in our work. Using the Lyapunov function they showed sufficient conditions for conditional global asymptotic stability for bounded solutions. The interesting point to observe is that in our setting, the same conditions that assure us of the existence of a solution and the existence of a unique stationary point are also sufficient for the existence of a solution that converges to the stationary point regardless of the initial conditions, i.e., they guarantee that the stationary point is conditionally globally asymptotically stable.

Thus, in terms of contribution to differential games we first provide a simpler proof for a setting similar to Scalzo. Then we extend this result by proving existence to the infinite horizon case. Third, because of our method of proof we are able to show the convergence to a stationary equilibrium regardless of the initial stocks of capital.

[2] Flaherty claims existence to her problem which is linear–quadratic. Her proof, however, is incomplete. The theorem in Pontryagin's book on which her existence claim is based is indeed *the* elementary theorem of differential equations that guarantees a solution in a small neighborhood around the initial conditions. An additional argument is needed to show continuation of the solution for the interval $(0, \infty)$.

2. Formulation

We consider a game G with two players where the payoff for each player is its total discounted profits. Instantaneous profits depend on the firm's own capital stock as well as the capital stocks of its rivals. Capital stock K_i accumulates according to the Nerlove–Arrow capital accumulation equation

$$\dot{K}_i = I_i - \delta_i K_i, \qquad K_i(0) = K_{i0}, \quad i = 1, 2. \tag{1}$$

Where I_i is the investment in the capital stock K_i of firm i, and δ_i is the depreciation constant. The planning horizon is denoted by T.

To define a game we have to specify the strategy spaces S_1, S_2 and the payoffs.

Player i's strategy is assumed to belong to the following set:

$$S_i = \{I_i(t)\colon [0, T] \to [0, \bar{I}_i] \mid I_i(t) \text{ is piecewise continuous on } [0, T]\},$$

where $\bar{I}_i$ is given in Assumption 1.

The payoff for firm i is defined by

$$J_i = \int_0^T e^{-rt}\{\pi_i(K_1, K_2) - C_i(I_i)\}\, dt, \tag{2}$$

where r is the discount rate, T might be finite or infinite, and $C_i(I_i)$ is the cost of investing I_i units.

Assumption 1. *The control $I_i(t)$ takes its value in a compact set $[0, \bar{I}_i]$. For example, a cost function $C_i(I_i)$ that is convex and satisfies that* $\lim C_i \to \infty$ *as $I_i \to \bar{I}_i$ will induce a control function satisfying Assumption* 1.

The instantaneous profit function $\pi_i(K_1, K_2)$ and cost function $C_i(I_i)$ satisfy

Assumption 2. *$\pi_i(K_1, K_2) \in C^2$, is increasing and strictly concave function of K_i, decreasing in K_j (for $i \neq j$, $i, j = 1, 2$), $C_i(I_i) \in C^2$, is strictly increasing strictly convex, and $C_i'(0) = 0$ (for $i = 1, 2$).*

It can be checked that the two examples given earlier can satisfy Assumption 2 (with respect to the revenue function). In Example 2 note that the revenue function will be increasing and concave in G_i if f is increasing and concave in its argument; see Fershtman [9].

We consider an open loop differential game, i.e., the problem of player i is to maximize (J_i) subject to his capital constraint given in (1), given $K_j(t)$ for $(j \neq i)$.

Define the game $G(K_{10}, K_{20}, T)$ as the game with strategy spaces S_i, payoff functions as in (2), time horizon T, and at $t = 0$, the game starts at the initial stocks of $K_i(0) = K_{i0}$ ($i = 1, 2$) and satisfies Assumptions 1 and 2. Finally, let $K_0 = (K_{10}, K_{20})$.

A *Nash Equilibrium* for the game $G(K_0, T)$ (for $T \in [0, \infty)$) is a pair of functions $I_1^*(t)$, $I_2^*(t)$ such that $I_i^*(t)$ maximizes (2) subject to (1) given $I_j^*(t)$ ($i \neq j$).

A *Stationary Nash Equilibrium* for $G(K_0, T)$ is a pair of values (I_1^*, K_1^*), (I_2^*, K_2^*) such that $I_i^* = \delta_i K_i^*$ and the pair (I_1^*, I_2^*) is a Nash equilibrium for the game $G(K_1^*, K_2^*, \infty)$.

We shall call a stationary equilibrium point (K_1^*, K_2^*) *conditionally locally asymptotically stable* if there exists a two-dimensional manifold S, containing $(K_1^*, K_2^*, I_1^*, I_2^*)$ such that for every $(K_1, K_2, I_1, I_2) \in S$ the solution of the game G which starts at (K_1, K_2, I_1, I_2) converges to the stationary equilibrium point.

We shall call a stationary equilibrium point (K_1^*, K_2^*) *conditionally globally asymptotically stable* if there exists a two dimensional manifold S, containing $(K_1^*, K_2^*, I_1^*, I_2^*)$ such that for every initial conditions K_{10}, K_{20} there exists a pair of initial investment I_{10}, I_{20} such that $(K_{10}, K_{20}, I_{10}, I_{20}) \in S$ and the solution of the game $G(K_{10}, K_{20}, \infty)$ converges to the stationary equilibrium point (K_1^*, K_2^*).

3. Finite Time Horizon

In this section we consider the game $G(K_{10}, K_{20}, T)$ for finite time horizon T.

We prove that for any K_0, and any T, there exists a path $(I_1(t), I_2(t))$, such that this pair of functions is a Nash equilibrium for the game G.

Define the following family of functions

$$B_{\mathrm{L}i}([0, T]) = \{f \in C([0, T]) \mid 0 \leqslant f(t) \leqslant \bar{I}_i/\delta_i \text{ and } |f(t) - f(s)| \leqslant \bar{I}_i |(t - s)| \text{ for all } t, s \in [0, T]\},$$

where $C([0, T])$ is the family of continuous, bounded function on $[0, T]$. Thus the family $B_{\mathrm{L}i}$ is bounded by a common bound and is "equiLipschitz," i.e., all the functions of the family share the same Lipschitz constant.

LEMMA 3.1. *$B_{\mathrm{L}i}([0, T])$ is a convex, compact subset of $C([0, T])$.*

Proof. We make use of Arzela Ascoli theorem (see Dunford and Schwartz, [8, Chap. 4] that states that if M is compact then a set in $C(M)$ is conditionally compact if and only if it is bounded and equicontinuous.

Let $M = [0, T]$ and let $C(M)$ be $B_{\mathrm{L}i}$. Since equiLipschitz implies equicontinuity of $B_{\mathrm{L}i}$, Arzela Ascoli theorem can be applied and so $B_{\mathrm{L}i}$ is conditionally compact for $i = 1, 2$.

Furthermore, by applying the triangle inequality it is clear that $B_{\mathrm{L}i}$ is closed since a converging sequence of equiLipschitz functions converges to a Lipschitz function with the same constant. Convexity can be shown in the same fashion.

For each strategy $I_i(t) \in S_i$ define the induced capital path as $K_i(t)$ which is the solution of Eq. (1). Assumption (1) guarantees that $I_i(t)$ is bounded by $\bar{I}_i$. Equation (1) guarantees that $K_i(t)$ is continuous and bounded by $\bar{K}_i = \bar{I}_i/\delta_i$ and that its Lipschitz coefficient is $\bar{I}_i$. Thus every induced capital path $K_i(t)$ is a member of $B_{\mathrm{L}i}([0, T])$. For every $K_j(t) \in B_{Lj}$ ([0, T]) consider the problem of maximizing (2) subject to (1) as a regular control problem for player i. Under Assumptions 1 and 2 (which guarantee sufficiency) there exists a unique $\tilde{I}_i(t)$ that solves this control problem (see, e.g., Lee and Markus, [16, Chap. 4] for finite time horizon and Baum [3] for the infinite case). Clearly $\tilde{I}_i$ induces a unique path of $\tilde{K}_i \in B_{\mathrm{L}i}([0, T])$.

ASSUMPTION 3. *$\pi_i^i = \partial\pi_i/\partial K_i$ is bounded, i.e., $|\pi_i^i| \leqslant L$ for some $L > 0$.*

LEMMA 3.2. *Consider a function $\phi_i: B_{Lj}([0, T]) \to B_{\mathrm{L}i}([0, T])$ such that $\phi_i(K_j(t)) = \tilde{K}_i(t)$. Under Assumptions 1–3, the function is continuous with respect to the supremum metric $\|f - g\| = \sup_t |f(t) - g(t)|$.*

Proof. We first consider the unconstrained maximization in which $I_i(t)$ is allowed to have negative values. What we show later on is that the optimal control is strictly positive for any t and thus the constrained and unconstrained maximization problems are equivalent. Consider the maximization problems for firm 1 in which the stock of player 2 is given by $K_2(t)$. The problem can be solved by using standard control theory.

Define the current value Hamiltonian to be

$$H_1 = \pi_1(K_1, K_2) - C_1(I_1) + \lambda_1 I_1 - \lambda_1 \delta_1 K_1.$$

Under Assumptions 1 and 2 the necessary conditions for optimality are sufficient as well since the Hamiltonian is concave in K_1 and I_1. The necessary conditions are

$$\dot{\lambda}_1 - r\lambda_1 = -\partial H_1/\partial K_1 = -\partial\pi_1/\partial K_1 + \lambda_1\delta_1, \tag{3}$$

$$\partial H_1/\partial I_1 = 0 = -C_1'(I_1) + \lambda_1. \tag{4}$$

The solution of Eq. (3) for λ is given by

$$\lambda(t) = \int^T \pi_1^1(K_1(s), K_2(s))\, e^{-(r+\delta_1)(s-t)}\, ds.$$

Since $\pi_1^1 > 0$, λ is strictly positive. Therefore our assumption on the cost function C and Eq. (4) guarantee that $I(t) > 0$. Solving Eq. (3) for λ, (4) for $I_1(t)$, substituting into (1) and solving for $K_1(t)$ yields

$$K_1(t) = \xi + \int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_S^T \pi_1^1(K_1(\tau), K_2(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds, \tag{5}$$

where $\xi = K_{10}\, e^{-\delta t}$, and π_1^1 denotes $\partial\pi_1/\partial K_1$.

We need to show that given a converging sequence $K_2^n(t) \to K_2^0(t)$, the corresponding sequence $K_1^n(t) = \phi_1(K_2^n(t))$ satisfies $K_1^n(t) \to K_1^0(t)$, where $K_1^0(t) = \phi_1(K_2^0(t))$.

Assume *a contrario* that $K_1^n(t)$ does not tend to $K_1^0(t)$. Without loss of generality (taking subsequence if necessary), we can assume that $K_1^n(t) \to J(t)$ but $J(t) \neq K_1^0(t)$. From the fact that B_{Li} is equiLipschitz, it follows that the convergence of K_i^n is uniform and thus this and the continuity of C_i' and π_1^1 imply that $J(t)$ satisfies

$$J(t) = \xi + \int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^T \pi_1^1(J(\tau), K_2^0(\tau))\, e^{-(r+\delta_1)(\tau-s)} d\tau \right\} ds.$$

Since the solution of (5) is unique, it follows that $J(t) = K_1^0(t)$. The fact that every converging sequence of K_1^n converges to K_1^0 implies that K_1^n tends to K_1^0.

Note that the functions ϕ_i are not reaction functions since they are not defined on the strategy space but rather on the state path space. If firm 2 chooses a path of investment $I_2(t)$ which induces a path of capital $K_2(t)$ then the optimal response of firm 1 will be to choose a path of investment such that the induced path of capital is $\phi_i(K_2(t))$.

THEOREM 1. *The differential game $G(K_{10}, K_{20}, T)$ associated with Eqs.* (1) *and* (2), *and satisfies Assumptions* 1–3 *has a Nash equilibrium solution for any initial conditions K_{10} and K_{20}.*

Proof. Define the function ϕ from $B_{L1} \times B_{L2}$ into itself as follows: For every $x \in B_{L1}$, $y \in B_{L2}$ let

$$\phi(x, y) = (\phi_1(y), \phi_2(x)). \tag{6}$$

We make use of the Schauder–Tychonoff theorem which states that if A is a compact convex subset of a locally convex linear topological space then every continuous mapping from A into itself has a fixed point.

Since $C([0, T])$ is a Banach space, from Lemma 2, $B_{L1} \times B_{L2}$ is a compact convex subset of a locally convex space, from Lemma 1 the function ϕ is a continuous mapping and thus ϕ has a fixed point. This fixed point is a Nash equilibrium solution for the game $G(K_0, T)$. Q.E.D.

The economic interpretation of Theorem 1 is that for every initial conditions K_{10} and K_{20}, there exists a pair of strategies $(I_1^*(t), I_2^*(t))$ such that: first, $I_i^*(t)$ is the best response for $I_j^*(t)$ and second, the induced capital paths $K_i^*(t)$ start at K_{i0}, for $i = 1, 2$.

In any such equilibrium, the assumptions of this model guarantee that both firms will be active. This can be seen by noting that $\pi_1^1(0, K_2) > (r + \delta_1) C_1'(0)$ and using Gould's argument. Moreover, the firm will find it optimal to invest at any time t. The formal argument is given in Lemma 3.2. The intuitive argument is as follows. Zero investment level cannot be optimal since the cost of investment $C'(0)$ is zero and the benefit from investing in capital is always positive, i.e., $\pi_i^i(K_i, K_j) > 0$.

4. Infinite Time Horizon

In this section we prove the existence of a Nash solution to the game $G(K_{10}, K_{20}, \infty)$ for every K_{10} and K_{20}. Replication of the finite time horizon proof is not possible. To see this note that we have defined a family of Lischitz functions $B_{\mathrm{Li}}([0, T])$. Then we defined mappings ϕ_i which, we were able to show, were continuous. Using this continuity and the compactness of $B_{\mathrm{Li}}([0, T])$ we were able to make use of the Tychonov theorem. In the infinite case, $B_{\mathrm{Li}}([0, \infty))$ is not compact. We therefore modify B_{Li} in a way to achieve compactness, and retain continuity.

Define the following family of functions

$$\Omega_{\mathrm{Li}}([0, \infty)) = \{f \in C([0, \infty)) \mid f = e^{-rt} g \text{ and } g \in B_{\mathrm{Li}}([0, \infty))\},$$

where $C([0, \infty))$ is the family of continuous, bounded functions on $[0, \infty)$.

LEMMA 4.1. *$\Omega_{\mathrm{Li}}([0, \infty))$ is a convex, compact subset of $C([0, \infty))$.*

Proof. We make use of an extension of the Arzela Ascoli theorem which states the following:

Let M be an arbitrary topological space and A a bounded subset of $C(M)$. Then A is conditionally compact if and only if for every $\varepsilon > 0$ there is a finite collection $E = \{E_1, ..., E_n\}$ of sets with union M and points $m_i \in E_i$

$i = 1, \ldots, n$, such that for $i = 1, \ldots, n$, $\text{Sup}_{f \in A} \text{Sup}_{m \in E_i} |f(m_i) - f(m)| < \varepsilon$ (see Dunford and Schwartz [8], Chap. 4.)

From the definition of $B_{\text{Li}}([0, T])$ (see Section 3) it is evident that due to the fact that B_{Li} is equiLipschitz, for every finite T there exists a collection E as required. Since the functions in $B_{\text{Li}}([0, \infty))$ are bounded by $\bar{K}_i$, for every given $\varepsilon > 0$, let T be such that $e^{-rT} 2\bar{K}_i < \varepsilon$. For this T define the collection E' as $\{E_1, \ldots, E_n, E_{n+1}\}$, where $E_{n+1} = [T, \infty)$. It is clear that, for $i = 1, \ldots, n+1$, and $m_i \in E_i$,

$$\underset{f \in \Omega_{\text{Li}}}{\text{Sup}} \ \underset{m \in E_i}{\text{Sup}} |f(m_i) - f(m)| < \varepsilon$$

and thus Ω_{Li} is conditionally compact. It is cumbersome but straightforward to check that Ω_{Li} is closed and thus it is compact.

Define a function $\phi_i : B_{Lj}([0, \infty)) \to B_{\text{Li}}([0, \infty))$ as the best induced capital path of player i for a given capital path of j as in Section 3. Define a function $\theta_i : \Omega_{Lj} \to \Omega_{\text{Li}}$ such that for every $f \in \Omega_{Lj}$

$$\theta_i(f) = e^{-rt} \phi_i(e^{rt} f). \tag{7}$$

The function θ_i is well defined since by definition of Ω_{Lj}, $e^{rt} f \in B_{Lj}$. In order to prove its continuity we need the following definition and lemma:

Definition. Let $x_n, x_0, \in B_{\text{Li}}([0, \infty))$. $x_n \to^* x_0$ iff for every finite T $\text{Sup}_{t \leqslant T} |x_n(t) - x_0(t)| \to 0$ as $n \to \infty$.

Lemma 4.2. *$e^{-rt} x_n \to e^{-rt} x_0$ iff $x_n \to^* x_0$.*

Proof. Clearly if $e^{-rt} x_n \to e^{-rt} x_0$ then for every finite T $\sup_{t \leqslant T} |e^{-rt} x_n - e^{-rt} x_0| \to_{n \to \infty} 0$ and thus $\sup_{0 \leqslant t \leqslant T} |x_n(t) - x_0(t)| \to_{n \to \infty} 0$. Conversely, since x_n are bounded for every given $\varepsilon > 0$, there is T_1 sufficiently large such that $\sup_{t \geqslant T_1} |e^{-rt} x_n - e^{-rt} x_0| < \varepsilon/2$. For sufficiently large n, $\text{Sup}_{t \leqslant T_1} |e^{-rt} x_n - e^{-rt} x_0| < \varepsilon/2$. Therefore for every $\varepsilon > 0$, there is T_1 and sufficiently large N such that for every $n \geqslant N$, $\sup_{t < \infty} |e^{-rt} x_n - e^{-rt} x_0| < \varepsilon$.

Assumption 4. *$|\pi_i^{ij}|$ is bounded, i.e., $|\pi_i^{ij}| \leqslant L_i$ for some $L_i > 0$ and C_i'' is bounded from below, i.e., $C_i'' > \varepsilon_i$ for some $\varepsilon_i > 0$.*

Lemma 4.3.[3] *Under Assumptions 1–4 the functions θ_i as defined in (7) are continuous with respect to the metric $\|f - g\| = \text{Sup}_t |f(t) - g(t)|$.*

Proof. Using Lemma 4.2 we need to show that given a converging sequence $K_2^n \to^* K_2^0$, the corresponding sequence $K_1^n = e^{rt} \theta_1(e^{-rt} K_2^n)$ satisfies $K_1^n \to^* K_1^0$, where $K_1^0 = e^{rt} \theta_1(e^{-rt} K_2^0)$.

[3] We are thankful to Dov Samet for pointing out this method of proof to us.

Without loss of generality, taking subsequences if necessary, we can assume that $K_1^n \to^* J$. We wish to show that $J = K_1^0$.

The solution of $K_1^0(t)$ following the procedure outlined in Lemma 2 is

$$K_1^0(t) = \xi + \int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^\infty \pi_1^1(K_1^0(\tau), K_2^0(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds. \tag{9}$$

Note that the inner infinite integral is bounded by $L/(r+\delta)$, where L is defined in Assumption 3.

Step 1. Observe the following expressions:

$$\int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^\infty \pi_1^1(J(\tau), K_2^n(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds, \tag{10}$$

$$\int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^\infty \pi_1^1(J(\tau), K_2^0(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds, \tag{10a}$$

where $J(\tau)$ is the value of the function J (the limit of K_1^n) at time τ. For a given t, the difference between (10) and (10a) tends to zero as $n \to \infty$.

This is true since, by Assumption 4, $[(C_1')^{-1}]'$ and π_1^{12} are bounded and so as $n \to \infty$,

$$\int_s^\infty |\pi_1^1(J(\tau), K_2^n(\tau)) - \pi_1^1(J(\tau), K_2^0(\tau))|\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \to 0.$$

Step 2. Define the following expressions:

$$J - \int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^\infty \pi_1^1(J(\tau), K_2^n(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds, \tag{11}$$

$$K_1^n - \int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^\infty \pi_1^1(K_1^n(\tau), K_2^n(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds. \tag{11a}$$

The difference between (11) and (11a) tends to zero as $n \to \infty$. This is true since $K_1^n \to^* J$ and by Assumption 4, $[(C_1')^{-1}]'$ and π_1^{11} are bounded. Since (11a) is identically zero, for a given t, by definition of K_1^n, it follows that expression (11) tends to zero when $n \to \infty$.

Step 3. The second term in expression (11) tends to J and to expression (10a) as $n \to \infty$. Thus

$$J = \int_0^t e^{-\delta_1(t-s)}(C_1')^{-1} \left\{ \int_s^\infty \pi_1^1(J(\tau), K_2^0(\tau))\, e^{-(r+\delta_1)(\tau-s)}\, d\tau \right\} ds$$

since the solution of (9) is unique, it follows that $J = K_1^0$.

THEOREM 2. *The differential game* $G(K_{10}, K_{20}, \infty)$ *satisfying Assumptions* 3 *and* 4 *has a Nash equilibrium solution for any initial conditions* K_{10} *and* K_{20}.

Proof. The proof follows the proof of Theorem 1, where $\Omega_{\mathrm{L}i}$, Lemmas 4.1 and 4.3 replace $B_{\mathrm{L}i}$, Lemma 3.1 and 3.2, respectively. Q.E.D.

5. STATIONARY EQUILIBRIUM AND ITS PROPERTIES

In this section we show the existence of a stationary equilibrium, discuss the concept of a Nash equilibrium manifold and investigate the properties of the stationary equilibrium.

PROPOSITION 5.1 (Existence). *Under Assumptions* 1 *and* 2 *there exists a stationary Nash equilibrium point* (K_1^*, K_2^*).

Proof. Consider the maximization problem for firm in which the stock K_j of firm j is constant, i.e., $K_j(t) = \tilde{K}_j$. This problem can be solved using standard control theory as follows:

The necessary conditions are

$$\dot{\lambda}_i - r\lambda_i = -\partial H_i/\partial K_i = -\partial \pi_i/\partial K_i + \lambda_i \delta_i, \tag{12}$$

$$\partial H_i/\partial I_i = 0 = -C_i'(I_i) + \lambda_i. \tag{13}$$

Differentiating Eq. (13) with respect to time, and substituting λ_i and $\dot{\lambda}_i$ from (12) and (13) yields the following equation

$$C_i'' \dot{I}_i = (r + \delta_i)\, C_i' - \pi_i^i(K_i, \tilde{K}_j), \tag{14}$$

where π_i^i denotes $\partial \pi_i/\partial K_i$.

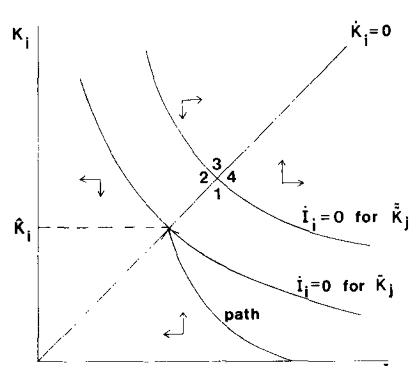

FIG. 1. $\dot{K}_i = 0$ is given by $I_i = \delta_i K_i$ and $\dot{I}_i = 0$ is given by $(r + \delta_i)\, C_i'(I_i) = \pi_i^i(K_i, \tilde{K}_j)$.

The solution to Eqs. (14) and (1) can be depicted on the (K_i, I_i) phase diagram. It is straightforward to check that the phase diagram is as in Fig. 1.

LEMMA 5.1. *There exists a unique intersection point between* $\dot{K}_i = 0$ *and* $\dot{I}_i = 0$, *and this intersection is a saddle point.*

Proof. The proof is straightforward. See, for example, Gould [13].

It follows that given $K_j(t) = \tilde{K}_j$ for any initial point $K_i(0)$ there exists a unique optimal path for firm i which converges to $\hat{K}_i$. $\hat{K}_i$ is thus the stationary optimal stock for firm i given $K_j(t) = \tilde{K}_j$.

The point at which both Eqs. (1) and (14) vanish yields an implicit equation for $\hat{K}_i$ as a function of $\tilde{K}_j$. This equation is given by

$$(r + \delta_i)\, C_i'(\delta_i \hat{K}_i) = \pi_i^i(\hat{K}_i, \tilde{K}_j). \tag{15}$$

Figure 1 depicts a case in which $\tilde{\tilde{K}}_j > \tilde{K}_j$ and $\pi_i^{12} = \partial^2 \pi_i / \partial K_1\, \partial K_2 > 0$, or the case where both inequalities are reversed.

Assumption 1 and Eq. (1) guarantee that $K_i(t)$ is bounded from above by $\bar{K}_i = \bar{I}_i / \delta_i$.

Define a function ϕ_i: $[0, \bar{K}_j] \to [0, \bar{K}_i]$ (for $i \neq j$, $i, j = 1, 2$) such that

$$\phi_i(\tilde{K}_j) = \hat{K}_i, \tag{16}$$

where ϕ_i is the solution of Eq. (15). Thus ϕ_i assigns for each constant level of $\tilde{K}_j$ the stationary solution of firm i. The continuity of C_i and π_i^i implies the continuity of the functions ϕ_i. Define a function ϕ from $[0, \bar{K}_1] \times [0, \bar{K}_2]$ to itself such that

$$\phi(K_1, K_2) = (\phi_1(K_2), \phi_2(K_1)); \tag{17}$$

ϕ is a continuous function from a compact convex set into itself, thus using Brouwer fixed point theorem there exists K_1^*, K_2^* such that

$$(K_1^*, K_2^*) = \phi(K_1^*, K_2^*) = (\phi_1(K_2^*), \phi_2(K_1^*)).$$

Thus $K^* = (K_1^*, K_2^*)$ satisfies the condition for a stationary Nash equilibrium point for the game G. Q.E.D.

Note that ϕ_i is *not* a best response or "reaction function." The function ϕ_i is not defined on the strategy space but rather on the state space. If firm 1 is at K_1^* for the rest of the game, only then the best strategy for player 2 is to converge to $\phi_2(K_1^*)$.

Let π^{ij} denote $\partial^2 \pi / \partial K_i\, \partial K_j$. The following assumption, in addition to Assumptions 1 and 2, are sufficient for uniqueness of the stationary equilibrium.

Assumption 5. *$\pi_i(K_1, K_2)$, $i = 1, 2$, satisfy the following inequality for all K_1 and K_2,*

$$\pi_1^{11}\pi_2^{22} > \pi_1^{12}\pi_2^{12}$$

and $\pi_i^{12} \neq 0$ for $i = 1, 2$, and all K_1 and K_2. Note that in the symmetric case when $\pi_1 = \pi_2 = \pi$, the assumption is a concavity assumption on π.

Proposition 5.2 (Uniqueness). *Under Assumptions* 1, 2, *and* 5 *the stationary equilibrium point is unique.*

Proof. Since $\pi_i^{ii} < 0$, the sign of ϕ_i' is the same as the sign of π_i^{12}. If π_1^{12} and π_2^{12} have opposite signs, the equilibrium point whose existence is guaranteed by Proposition 2 is necessarily unique.

If $\pi_i^{12} > 0$ for $i = 1, 2$, then it is sufficient to prove that at any equilibrium point $(\phi_1^{-1})' > \phi_2'$. Since ϕ is the solution of (15), this last condition is equivalent to the following condition:

$$(\delta_1(r+\delta_1)\, C_1'' - \pi_1^{11})(\delta_2(r+\delta_2)\, C_2'' - \pi_2^{22}) > \pi_1^{12}\pi_2^{12}. \tag{18}$$

If, however, $\pi_i^{12} < 0$ for $i = 1, 2$, then it suffices to show that $(\phi_1^{-1})' < \phi_2'$. As before, this is equivalent to condition (18). Since Assumption 5 holds, then necessary (18) holds and the equilibrium point is unique. Q.E.D.

Proposition 5.3 (Conditional local stability). *Under assumption* 1, 2, *and* 5 *the stationary equilibrium point is conditionally locally asymptotically stable.*

Proof. What we need to show is that the Jacobian matrix of the following system has two positive and two negative (real parts of the) eigenvalues at the equilibrium point.

$$\dot{K}_1 = I_1 - \delta_1 K_1, \tag{19}$$

$$\dot{K}_2 = I_2 - \delta_2 K_2, \tag{19a}$$

$$C_1''\dot{I}_1 = (r+\delta_1)\, C_1' - \pi_1^1(K_1, K_2), \tag{19b}$$

$$C_2''\dot{I}_2 = (r+\delta_2)\, C_2' - \pi_2^2(K_1, K_2). \tag{19c}$$

If Δ is an eigenvalue, it is straightforward to check that Δ has to satisfy the following condition:

$$f(\Delta) = \pi_1^{12}\pi_2^{12}/C_1''C_2''. \tag{20}$$

Where $f(\Delta) = f_1(\Delta) f_2(\Delta)$ and $f_i(\Delta)$ is given by

$$f_i(\Delta) = (r+\delta_i - \Delta)(\delta_i + \Delta) - \pi_i^{ii}/C_i''. \tag{21}$$

It is clear that $\lim_{\Delta \to \pm\infty} f_i(\Delta) = -\infty$ and thus $\lim_{\Delta \to \pm\infty} f(\Delta) = \infty$. In addition $f(\Delta)$ achieves a local maximum at $\Delta = r/2$, and the equation $f_i(\Delta) = 0$ has two real roots, one positive and one negative. The function $f(\Delta)$ has one maximum at positive Δ and two minima, one at positive Δ and another negative. A necessary and sufficient condition for Eq. (20) to have two positive and two negative roots is that $f(0) > \pi_1^{12}\pi_2^{12}/C_1''C_2''$. This condition is exactly Eq. (18) which holds if Assumption 5 is valid.

From a well-known theorem of differential equations there exists a two-dimensional manifold S such that the solution of Eqs. (1) and (14) starting on the manifold, converges to the equilibrium point. See, for example, Coddington and Levinson [7, Chap. 13]. Q.E.D.

Define the set of $K(S)$ as the following projection of S, i.e.,

$$K(S) = \{K \in \{[0, \bar{K}_1] \times [0, \bar{K}_2]\} \mid \text{there exists } I = (I_1, I_2) \text{ such that } (K, I) \in S\}.$$

We now have the following corollary: For every initial condition $K_0 \in K(S)$, the game $G(K_0, \infty)$ has a solution which converges to the stationary equilibrium point. To see this, note that by definition of $K(S)$, for K_0 there exists a pair $I_0 = (I_1(0), I_2(0))$ such that $(K_0, I_0) \in S$ and therefore there exists a unique path which starts at (K_0, I_0) and ends at (K^*, I^*). Since along this path conditions (1) and (14) are satisfied for $i = 1, 2$ we only have to show that the transversality conditions are satisfied. It will then follow that $I_i(t)$ is the best response for $I_j(t)$ since Assumptions 1 and 2 guarantee the sufficiency of the necessary conditions.

The transversality condition for control problems with infinite horizons that were proven by Michel [18] are that the discounted Hamiltonian vanishes as t approaches infinity. This is satisfied in our case since the instantaneous profit function is bounded and at the stationary equilibrium $I_i^* = \delta_i K_i^*$ and λ is bounded.

Thus the manifold S can be described as a *Nash equilibrium manifold* since for any initial condition K_0 in its projection there exist I_0 such that there exists a Nash solution to the game that lies on the manifold and converges to a steady state.

In the next section, we investigate the spanning range of this manifold (or its continuation).

6. Characterization and Convergence of the Nash Solution

In this section we investigate the properties of the Nash solution. In particular we examine its convergence properties.

The analysis involves phase diagrams where the boundaries are nonstationary. For the pioneering work on this subject see Kamien and Schwartz [15]. For further work on this subject see Muller [19].

Consider Fig. 1 which depicts the (K_i, I_i) phase diagram. Define the movement of $\dot{I}_i = 0$ from $\tilde{K}_j$ to $\tilde{\tilde{K}}_j$ as "up." Whether the $\dot{I}_i = 0$ boundary moves up or its reverse (down) depends on the cross partial derivative of the revenue function, and on the sign of $\dot{K}_j$.

LEMMA 6.1. *Consider the game $G(K_{10}, K_{20}, \infty)$ satisfying Assumptions 3–5 and the function ϕ_i as defined in Section 4. If $\lim_{t\to\infty} K_j(t) = K_j^*$ and $K_i(t) = \phi_i(K_j(t))$ then $\lim_{t\to\infty} K_i(t) = K_i^*$, where (K_1^*, K_2^*) is the unique stationary equilibrium point.*

Proof. What we have to show is that if one player converges to the stationary equilibrium point then the induced capital path of the best response of the second player will converge as well.

Step 1. Assume there exists a finite T such that $K_j(t)$ is monotonic on $[T, \infty)$. We first claim that $K_i(t)$ is either monotonic or single peaked on $[T, \infty)$. Consider Fig. 2. For the path $K_i(t)$ to have an extremum point, it has to cross the $\dot{K}_i = 0$ line. Without loss of generality, we assume that the path is currently in region 2. Crossing from region 2 to region 1 is impossible. The only way to cross the $\dot{K}_i = 0$ line is for the $\dot{I}_i = 0$ boundary to move "down" and to catch up with the path. Once it crosses the path, the latter is in region 3 and it can now cross the $\dot{K}_i = 0$ line to region 4. The path is now depicted in Fig. 3. We now have to show that the path cannot have another extremum unless $\dot{K}_j$ changes sign which cannot happen by our monotonicity assumption of step 1. The path cannot cross from region 4 to region 3, therefore the $\dot{I}_i = 0$ boundary which has moved down has to change its direction, catch up with the path so that the path will be again in region 1

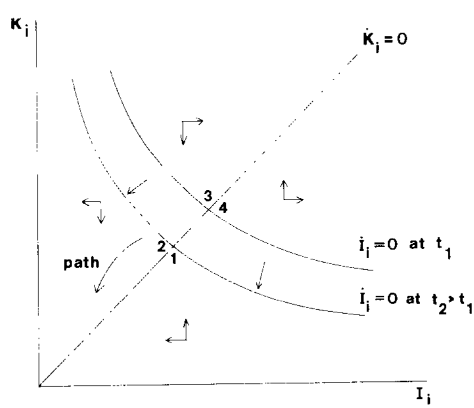

FIGURE 2

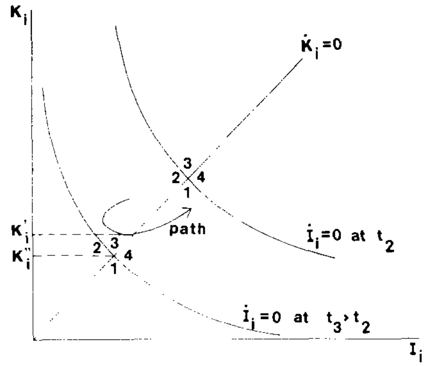

FIGURE 3

and the intersection with $\dot{K}_i = 0$ will be made possible. The boundary $\dot{I}_i = 0$ is given by $(r + \delta_i)\, C_i'(I) = \pi_i^i(K_i, K_j)$. It can change direction only if $\dot{K}_j$ changes sign. Since our assumptions rule out going out of business (see our discussion in Secton 3) the path K_i does not tend to zero. By standard arguments (see, e.g., Gould) the path K_i does not tend to infinity. Thus it converges to a stationary equilibrium point. Its uniquenes guarantees that $K_i(t)$ will converge to K_i^*.

Step 2. Assume that there does not exist a finite T such that $K_j(t)$ is monotonic on $[T, \infty)$. Thus for any t, there exists $\tau > T$ such that $K_j(\tau)$ is an extremum point.

For a given path $K_j(t)$, $j = 1, 2$, define a *cycle* as the path of $K_j(t)$ between two consecutive extremal points. Let the *amplitude* of a cycle be the difference between the maximum and the minimum of $K_j(t)$ in the cycle. Note that the arguments in step 1 can now be applied to any cycle of K_i, i.e., $\dot{K}_i$ cannot change sign more than once without $\dot{K}_j$ changing sign at least once. Since $K_j(t)$ tends to K_j^*, the amplitude of its cycles tend to zero as time tends to infinity. Using the phase diagram of Fig. 3, let $\hat{K}_i$ be the level of capital at the intersection of the curve $\dot{I}_i = 0$ and the line $\dot{K}_i = 0$. Changes in K_j will induce changes in the $\dot{I}_i = 0$ and therefore in $\hat{K}_i$. Define the cycles and amplitude of $\hat{K}_i$ as before. From Eq. (15) which describes $\hat{K}_i$ as a function of K_j and the assumption that $|\pi_i^{ij}|$ is bounded, we conclude that since the amplitudes of the cycles of K_j tends to zero, so do the amplitudes of $\hat{K}_i$ as time tends to infinity. In Fig. 3, let $\tilde{K}_i$ denote the level of capital when K_i achieves a minimum as depicted. Observe that necessarily $\hat{K}_i$ is smaller than $\tilde{K}_i$, since intersection cannot occur between zones two and one. Thus the $\dot{I}_i = 0$ curve has to be *below* the path for intersection to occur. Thus, the amplitudes of the cycles of K_i can be bounded using the amplitudes of the cycles of $\hat{K}_i$, which tend to zero. Thus, the amplitudes of the cycles of $K_i(t)$ tend to zero as well. The uniqueness of the stationary point guarantees that $K_i(t)$ tends to K_i^*.

THEOREM 3. *The differential game* $G(K_{10}, K_{20}, \infty)$ *satisfying Assumptions* 3–5 *has a Nash equilibrium solution that converges to the stationary equilibrium point for every initial codition* K_{10} *and* K_{20}.

Proof. Let

$$B_{Li}^* = \{f \in B_{Li}([0, \infty)) \mid \lim_{t \to \infty} f(t) = K_i^*\}.$$

Lemma 6.1 assures us that the range of the function ϕ_i is B_{Li}^*. In the same fashion we can define Ω_i^* (as in Section 4). It can be verified that Ω_i^* is conditionally compact (since it is a subset of a conditionally compact set) is closed and convex. Thus we can make use of the Schauder–Tychonov fixed-point theorem. Q.E.D.

The existence of the stationary manifold S guarantees that the only convergence to K_1^* and K_2^* is through the manifold. Thus a corollary of Theorem 3 is that if S' is the continuation of S on the (K_1, K_2) plane, S' spans the entire (K_1, K_2) plane. We thus have

COROLLARY. *The stationary equilibrium point* (K_1^*, K_2^*) *is conditionally globally asymptotically stable.*

The proofs of existence and global stability can be extended to the multifirm case. The extension to the multicapital case is considerably more difficult. In addition, we have shown existence only. Thus, it is possible that other Nash equilibria which are not globally stable exist as well.

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