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INTEGRAL GAMES: THEORY AND APPLICATIONS

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Abstract
The role of history in determining market process is widely acknowledged by economists. History can affect the market interaction in different ways. It can affect agents' behavior as well as the evolution of the type of market interaction over time. Differential games is a class of continuous time dynamic games in which players interact along time and the game is structurally dynamic. What characterizes differential games is that the transition of the state variables is determined only by the current state and the action taken by players and it is independent of the history leading to this state. In this paper we would like to extend the framework of differential games and introduce and discuss a class of dynamic games which we will denote as integral games. In this class of games the evolution of the state variable is given by a nondegenerate integral equation and thus the framework is capable of modeling a problem in which the whole history affects the changes of the state variables.

1. INTRODUCTION

The role of history in determining market process is widely acknowledged by economists. In a recent interesting paper David (1988) discussed the different aspects of the path dependent process and the role of history in economic analysis. History can affect the market interaction in different ways. It can affect agents' behavior as well as the evolution of the type of market interaction over time (for a discussion see also Kreps and Spence (1986)). Specifically, firms as well as other economic agents can condition their behavior on the history of the interaction. Discussing the market interaction as a repeated game gives rise to many interesting equilibria in which firms use history dependent strategies. Such a dependence enables players to use trigger strategies or to credibly threaten one another. At the same time the history may affect the evolution of the market game such that it determines the agents' payoff function as well as their available strategies. This type of effect is usually captured by state variables the evolution of which is determined by the players' actions. A useful framework for analyzing such a dynamic interaction is the dynamic games framework.

Differential games is a class of continuous time dynamic games in which players interact along time.[1] The game is structurally dynamic as there are state variables, which are changed along time according to some evolution functions, and the players' payoffs depend on these state variables as well as on their own actions. What characterizes differential games is that the evolution function is given by some differential equations that specify the changes of the state variables as a function of time, the players' choice of control and the value of the state variables at that time. Such a framework limits the discussion to economic problems that have a Markovian structure, i.e., the transition of the state is determined only by the current state and the actions taken by players and it is independent of the history leading to this state. Clearly, assuming such a structure might be restrictive when we come to model dynamic economic phenomena. In many interesting economic problems, the evolution of the state variables depends on the whole history of the game. As an example we can consider a model of capital accumulation game in which the depreciation rate might depend on the age of the capital or the machines the firms accumulate over time. Another example can be a learning-by-doing model in which the effect of production on experience clearly depends on the date at which the production took place. The depreciation cannot capture this effort as a constant depreciation rate implies that both recent experience and ones that accumulated in the past are depreciated at the same rate.

In this paper we would like to extend the framework of differential games and introduce and discuss a class of dynamic games which we will denote as *interal games*. In this class of games the evolution of the state variable is given by a nondegenerate integral equation and thus the framework is capable of modeling a problem in which the whole history affects the changes of the state variables. Clearly one can view differential games as a special class of integral to differential equations. One can expect that such a generalization introduces additional complexity into the analysis. We thus discuss in the paper the various problems that arise in extending the standard differential game framework to integral games and in particular the problematic extension of the standard strategy spaces (or information patterns). We present the necessary conditions that the open loop Nash equilibrium strategies satisfy and discuss the classes of game in which the open loop Nash equilibrium is also a subgame perfect Nash equilibrium.

We emphasize that this is only the first step in analyzing the integral games' framework and, indeed, there is more work that must be done in order to fully understand the type of equilibria that can arise in such a framework. We surmise, however, that this framework can capture more completely the role of history in dynamic interactions and thus can be helpful in modeling such phenomena.

2. A SIMPLE EXAMPLE OF NONDEGENERATE HISTORY DEPENDENT EVOLUTION: THE CASE OF CAPITAL ACCUMULATION AS AN INTEGRAL GAME

Before defining integral games we would like to give a short example that demonstrates the type of problems that can be handled by the use of integral games.

A capital accumulation game is a class of dynamic games in which each player accumulates some form of capital according to

$$dx_i/dt = \dot{x}_i = u_i(t) - \delta_i x_i(t), \; ; x_i(0) = x_{i0}; \; i = 1,\dots,n \tag{1}$$

where $x_i(t)$ is the capital of player i at time t, δ_i is a constant history independent depreciation rate and $u_i(t)$ is the investment of player i at time t. The players' payoff in this class of games is

$$J^i = \int_0^\infty F_i(x(t),u(t),t)dt \qquad i = 1,\dots,n \tag{2}$$

where $x(t) = (x_1(t), x_2(t),\dots,x_n(t))$ and $u(t) = (u_1(t), u_2(t),\dots,u_n(t))$. For more details about investment and capital accumulation games see Spence (1979), Fudenberg and Tirole (1985), Fershtman and Muller (1984, 1986), Reynolds (1987), Leitman and Schmitendorf (1978), Dockner and Jorgensen (1984), Feichtinger (1983), and Dockner (1984). The two latter models are general differential games for which open loop and closed loop equilibria coincide. For more details about differential games and solution concepts see Reinganum (1982), and Dockner, Feichtinger and Jorgensen (1985).

Characteristic to this class of games is that the control chosen by player j does not affect the payoffs of player i, nor the changes of x_j, $j \neq i$, i.e., each player is independently accumulating a capital.

Equation (1), which governs the evolution of the state variables along time, is equivalent to the following integral equation:

$$x_i(t) = x_i(0) + \int_0^t e^{-\delta_i(t-s)} u_i(s)ds \tag{3}$$

In assuming a differential game framework we limit our discussion only to Markovian structure, i.e., the evolution of the state variables (1) depends only on the actions of the players at time t and the state variables at time t. This assumption implies, for example, that in the capital accumulaltion problem the depreciation of the capital (see equations (1) and (3)) does not depend on the date at which this capital was constructed. Specifically, let us define the stock of capital as follows:

$$x_i(t) = x_i(0) + \int_0^t u_i(s) e^{-\delta_i(s)(t-s)} ds \; . \tag{4}$$

Equation (4) is an integral equation that cannot be reduced to a differential equation. It allows us to model situation in which "old" capital has a higher depreciation rate than a "new" capital. Such a situation, as was stated above, cannot be discussed as a differential game, as Eq. (4) cannot be reduced to a differential equation, and thus calls for the development of a more general framework. In a monopoly market structure this capital accumulation problem was investigated in Muller and Peles (1990).

3. NOTATIONS AND DEFINITIONS

3.1. Definitions of Integral Games

We define an integral game as a class of dynamic games (N, U, X, J, I, z) such that:

(i) $N = \{1,...,n\}$ is a set of players.

(ii) $U = (U_1,...,U_n)$; U_i is a set of admissible controls such that at every t player i chooses $u_i \in U_i$.[2]

(iii) $X \in R^m$ is a set of all possible vectors of state variables.

(iv) $J = (J_1,...,J_n)$ where J_i is the objective function of player i. Specifically, we let

$$J_i = \int_0^T F_i(t,x(t),u(t))dt \qquad i = 1,...,n \tag{5}$$

where $x(t) = (x_1(t),...,x_m(t))$ and $u(t) = (u_1(t),...,u_n(t))$.

(v) $I = (I_1,...,I_m)$, where I_j is an integral equations that determine the value of x_j at any specific date. Specifically, I_j will be defined by the following equation:

$$x_j(t) = x_{j0} + \int_0^t f_j(t,x(s),u(s),s)ds \qquad j = 1,...,m \; . \tag{6}$$

Note that formally differentiated games can be considered as a subclass of integral games. This is the case when the integral equation (6) can be reduced to a differential equation. Such an integral equation is reducible if, for example, $\partial f_j/\partial t = 0$, i.e., f does not depend on t, or in the general case when $df_j/dt = c(t)\, f_j(x(s), u(s), s)$ for some function c(t). In the general case, however, (v) implies that the evolution of the state variable might depend on the whole history of the game and is not reducible.

There are several standard types of information patterns, $z_i(t)$, in dynamic games literature (see Basar and Olsder (1982)).

$z_i(t) = \{x_0\}$: open loop
$z_i(t) = \{x(t)\}$: feedback
$z_i(t) = \{x(\tau), \tau \in [0, t]\}$ in which the players know the whole history. Sometimes this is referred to as closed loop (with memory).

In principle we can extend the use of these information patters to integral games. However, such an extension needs some further elaboration before being applied.

3.2. Open Loop Strategies

In a differential game setting the open loop strategy is a time path of actions $u_i(t)$ that specifies the i'th player's actions at every t. Each player is fully committed to this path at the outset of the game. Clearly one can discuss integral games with open loop strategies. Formally, one can view the open loop strategies as function that assigns for every initial condition the whole path of control. Thus, in extending the open loop strategies to integral games, one needs to re-examine the meaning of initial conditions. By saying that the information set is $\{x_0\}$ we mean that the players can observe the state of the system at time zero. Observe also that in our definition of integral games equation (6) which specifies the state of the system at period t does not depend on the history prior to time zero or more specifically on the actions leading to the initial state x_0. This is, however, only a special case. The importance of the way we define and incorporate the initial state becomes obvious once we discuss the time consistency issue.

A well-known and trivial observation is that in differential games open loop Nash equilibrium is time consistent. That is to say that if we ask the players, at some intermediate time, whether they wish to revise their strategies they will stick to the original equilibrium strategies. The reason for this is that the truncated strategies consitute an open loop Nash equilibrium in the game that starts at that particular intermediate point. Discussing this time consistency property for the open loop equilibrium of integral games gives rise to the following technical problem: even if at time zero the only relevant information is x(0) still if we look at the game that starts at some intermediate point y and discussing the open loop Nash equilibrium of the game, some difficulty arises as the players cannot rely only on the knowledge of the state of the system at the intermediate point. They need to know the path of actions between time zero and the intermediate point in order to use equation (6) and calculate the state of the system at some future periods. Thus, when discussing integral games one needs to modify the definition of open loop strategies.

Let h_t be the history prior to time t. An open loop strategy in integral games is a path $u_i(h_0, t)$. The player conditions his actions on the history prior to the outset of the game and the value of the state variable at the outset. We later on specify the necessary and sufficient conditions for such open loop Nash equilibrium.

3.3. Feedback Strategies

The feedback Nash equilibrium is the one most commonly used in economic applications. Each player in this case adopts a decision rule that specifies his action at time t as a function of the state variables at that time. The use of such strategies is intuitive once one has a Markovian structure. Moreover, in a differential game setting, if one player uses a feedback decision rule, his competitor cannot gain anything if he is in possession of information regarding the specific history of the game or if he conditions his actions on this knowledge. Thus the feedback Nash equilibrium is in particular also an equilibrium when we consider history dependent strategies.

In the integral game framework, letting the players have feedback information patterns implies that although players know the state of the world they do not know how it will evolve over time, since this evolution depends on the history of the game which is not known to them. Clearly one can formally define feedback equilibrium in integral games, but then one should consider a game with incomplete information since each player does not know the type of his competitor and moreover his own type when the type is defined by the past actions of each player.

3.4. Closed Loop (History Dependent) Strategies

Let h_t be the history up to period t. A history consists of control combination $\{u_i(\tau) \mid \tau \in [0, t]\}$, i = 1,2. We let H_t denote the set of all possible histories of length t. A closed loop strategy is a function that prescribe a control for every history $h_t \in H_t$. Thus, the closed loop strategies space can thus be considered as the history dependent strategies space.

Clearly, most economic problems should be analyzed using the closed loop (history dependent) strategy space. And such an analysis is an essential part of applying integral games in economic analysis. But such an analysis is beyond the scope of this paper and is part of our future research agenda.

3.5. Classes of Integral Games for which the Open Loop Nash Equilibrium is a Special Case of Closed Loop (History Dependent) Strategies

As the subgame perfect feedback Nash equilibrium in a differential game is in most cases not tractable there has been extensive research identifying the classes of games for which the open loop Nash equilibrium is also a feedback equilibrium (see, for example, Mehlman and Willig (1985), Reinganum (1982), and Dockner, Feichtinger and Jorgensen (1985)).

In discussing integral games one can discuss the classes of games for which the open loop Nash equilibrium is also a (subgame perfect) closed loop Nash equilibrium.

Fershtman (1987) identified the necessary and sufficient conditions for the above equivalence for differential games. Specifically, this condition implies that once the open loop equilibrium is invariant under changes of the initial conditions, then the open loop is also a closed loop equilibrium. One can extend these conditions to integral games and use them for the identification of subgame perfect open loop Nash equilibria. As an example, let us consider an exponential integral game denoted by G_1 such that

$$J_i = \int_0^T [e^{-rt} e^{\gamma x_i(t)} F(u) dt$$

where x_i (t) is a state variable and u is a vector of control determined by the following integral equation:

$$x_i(t) = x_i(0) + \int_0^t f(t,u(s),s)ds, \quad i = 1,\ldots,n .$$

Let us use the state transformation y = x-x(0), and let $A_i = e^{\gamma x_i(0)}$. The game G_1 is now equivalent to the following game (G_2):

$$J_i = A_i \int_0^T e^{-ert} e^{\gamma y_i(t)} F(u) dt$$

subject to

$$y_i(t) = \int_0^t f(t,u(s),s)ds, \quad i = 1,\ldots,n .$$

Clearly the solutions are invariant under multiplications of the payoff function by a constant. Thus, for this class of games the open loop Nash equilibrium is invariant under changes of the initial conditions as every open loop Nash equilibrium for the game G_1 with the initial condition x(0) is also an open loop Nash equilibrium for the game G_2 that is identical to G_1 but for the initial condition x(0) = 0.

This implies that for this class of integral games, the open loop Nash equilibrium is also a closed loop history dependent Nash equilibrium.

4. NECESSARY CONDITIONS: OPEN LOOP

For the sake of tractability we consider in this section only games for which m = n (the case of m < n can be followed immediately).

A Nash equilibrium for the above game with path strategies (open-loop) can be described with the help of the Hamiltonian of player i:

$$H_i(t,x(t),u(t),\lambda(t)) = F_i(t,x(t),u(t)) + \sum_{j=1}^{n} \int_t^T f_j(s,x(t),u(t),t)\lambda_{ij}(s)ds \tag{7}$$

where λ_{ij} is the evaluation of state variables x_j by player i. We can now formally prove the following theorem (the proof is along the lines of the variational approach employed by Smith (1974, pp. 288-293)).

Theorem 1: If there exists a solution to the integral game stated above, and F_j and f_j are continuous, with continuous partial derivatives and $f_j(t, x, u, s) = 0$ for $t < s$, then there exist continuous multiplier functions λ_{ij} and Hamiltonians H_i defined by (7) that satisfy the following conditions:

$$\partial H_i/\partial u_i = 0 = \partial F_i/\partial u_i + \sum_{j=1}^{n} \int_t^T \partial f_j(s,x(t),u(t),t)/\partial u_i \lambda_{ij}(s)ds, \qquad 1 \le i \le n \tag{8a}$$

$$\partial H_i/\partial x_j = \lambda_{ij} = \partial F_i/\partial x_j + \sum_{k=1}^{n} \int_t^T [\partial f_k(s,x(t),u(t),t)/\partial x_j]\lambda_{ik}(s)ds, \qquad 1 \le i, j \le n \tag{8b}$$

Note that we do not need the transversality condition in order to prove the above conditions.

Proof: Let u(t) be a fixed continuous control control function. Then the solution to (6), assumed to exist, can be written as:

$$x = x(t,u) \tag{9}$$

where $x = (x_1,...,x_n)$ and $u = (u_1,...,u_n)$, and substituted into (6) to yield a functional in u,

$$J(u) = \int_0^T F(t,x(t,u),u)dt . \tag{10}$$

Let variation in u_i be denoted by $\epsilon\Delta u_i$ where ϵ is an arbitrary small number and $\Delta u_i(t)$ is a continuous function in t. Let $\Delta u = (0,...,0,\Delta u_i, 0,...,0)$ and therefore:

$$u + \epsilon\Delta u = (u_1(t),...,u_{i-1}(t),u_i(t) + \epsilon\Delta u_i(t),u_{i+1}(t),...,u_n(t)) .$$

Since u(t) constitutes the control function of a Nash equilibrium, the variation of J, written

$$\delta J(u,\Delta u) = (d/d\epsilon)J(u + \epsilon\Delta u)|_{\epsilon=0} = 0 \tag{11}$$

vanishes when $\epsilon = 0$ (see Smith, 1974). Let F_{ix} be the vector whose j^{th} component is $\partial F_i/\partial x_j$. Computation of the variation of (10) and combination with (11) yields:

$$\int_0^T [F_{ix}(t,x(t,u)\ \partial/\partial\epsilon x(t,u + \epsilon\Delta u)|_{\epsilon=0} + \partial F_i/\partial u_i(t,x(t,u),u)\Delta u_i]dt \equiv 0 . \tag{12}$$

Now from (6):

$$\begin{aligned}\partial/\partial\epsilon x_j(t,u+\epsilon\Delta u)|_{\epsilon=0} &= \partial/\partial\epsilon \int_0^t f_j(t,x(s,u+\epsilon\Delta u),u + \epsilon\Delta u,s)ds|_{\epsilon=0} \\ &= \int_0^t \partial/\partial\epsilon f_j(t,x(s,u+\epsilon\Delta u),u+\epsilon\Delta u,s)|_{\epsilon=0}ds \\ &= \sum_{k=1}^{n} \int_0^t [\partial f_j/\partial x_k(t,x(s,u),u,s)\ \partial/\partial\epsilon x_j(s,u+\epsilon\Delta u_i)|_{\epsilon=0} \\ &\quad + \partial f_j/\partial u_i(t,x(s,u)u,s)\Delta u_i]ds .\end{aligned} \tag{13}$$

Let A(t, s) be the matrix whose (j, k) element is $\partial f_j/\partial x_k$.
Let B(t, s) be the vector whose j^{th} element is $\partial f_j/\partial u_i$. (14)
Let y(t) be the vector whose j^{th} element is: $\partial/\partial\epsilon\, x_j(t, u + \epsilon\Delta u)\,|_{\epsilon=0}$.

In a matrix form the system of equations (13) for j = 1,...,n can be written as the following linear integral equation:

$$y(t) = \int_0^t [A(t,s)y(s) + B(t,s)\Delta u_i(s)]ds \,. \tag{15}$$

According to a basic property of linear integral equations (see Miller 1971, pp. 189-93), the solution to (15) can be written as:

$$y(t) = \int_0^t r(t,s)[\int_0^s B(s,\tau)\Delta u(\tau)d\tau]ds + \int_0^t B(t,s)\Delta u_i(s)ds \tag{16}$$

where the matrix function r(t, s), called the resolvent kernel, satisfies

$$r(t,s) = A(t,s) + \int_s^t r(t,\tau)A(\tau,s)d\tau \,. \tag{17}$$

Now, on a triangular region $0 \le \tau \le t \le T$, and function g

$$\int_0^T \int_0^t g(t,\tau)d\tau dt = \int_0^T \int_t^T g(\tau,t)d\tau dt \tag{18}$$

so (16) can be rewritten as

$$y(t) = \int_0^t [B(t,s) + \int_s^t r(t,\tau)B(\tau,s)d\tau]\Delta u_i(s)ds \,. \tag{19}$$

Recalling (14) and substituting from (19) into (12) yields:

$$\int_0^T \{F_{ix}(t,x(t,u),u(t))[\int_0^t [B(t,x) + \int_s^t r(t,\tau)B(\tau,s)d\tau]\Delta u_i(s)ds] \\ + \partial F_i/\partial u_i(t,x(t,u),u(t))\Delta u_i(t)\}dt = 0 \,. \tag{20}$$

Application of (18) to (20) yields

$$\int_0^T \{\int_t^T F_{ix}(s,x(s,u),u(s))[B(s,t) + \int_t^s r(s,\tau)B(\tau,t)d\tau]ds \\ + \partial F_i/\partial u_i(t,x(t,u),u(t))\}\Delta u_i(t)dt = 0 \,. \tag{21}$$

Since (21) must hold for all continuous functions $\Delta u_i(t)$, it must obtain in the particular case when $\Delta u_i(t)$ equals the curly bracketed expression in (21). This implies

$$\int_0^T \{\int_t^T F_{ix}(s,x(s,u),u(s))[B(s,t) + \int_t^s r(s,\tau)B(\tau,t)d\tau]ds \\ + \partial F_i/\partial u_i(t,x(t,u)(t))\}^2 dt = 0 \,. \tag{22}$$

from which it follows that

$$\int_t^T F_{ix}(s,x(s,u),u(s))[B(s,t) + \int_t^s r(s,\tau)B(\tau,t)d\tau]ds \\ + \partial F_i/\partial u_i(t,x(t,u),u(t)) = 0 \,. \tag{23}$$

Another application of (18), this time to (23), yields a Euler equation

$$\int_t^T [F_{ix}(s,x(s,u),u(s)) + \int_s^T F_{ix}(\tau,x(\tau,u),u(\tau))r(\tau,s)d\tau]B(s,t)ds \\ + \partial F_i/\partial u_i(t,x(t,u),u(t)) = 0 \,. \tag{24}$$

If we can define $\lambda(s)$ by

$$\lambda(s) = F_{ix}(s,x(s,u),u(s)) + \int_s^T F_{ix}(\tau,x(\tau,u),u(\tau))r(\tau,s)d\tau \tag{25}$$

and substitute from (25) into (24) we get

$$\partial F_i/\partial u_i(t,x(t,u),u(t)) + \int_t^T \lambda(s)B(s,t)ds = 0 . \tag{26}$$

Substitution from (14) and recollection of (8) discloses that

$$\partial F_i/\partial u_i(t,x(t,u),u(t)) + \sum_{j=1}^{n} \int_t^T \partial f_j/\partial u_i(s,x,u,t)\lambda_{ij}(s)ds = 0 = \partial H_i/\partial u_i \tag{27}$$

verifying equation (8a).

To establish equation (8b) it has to be shown that the $\lambda(t)$ defined in (25) is consistent with its definition given in equation (8b). Substituting (14) into (8b) yields:

$$\lambda(t) = F_{i=x}(t,x(t,u),u(t)) + \int_t^T A(s,t)\lambda(s)ds, \tag{28}$$

a linear integral equation in λ. Consistency of (25) and (28) reduces to demonstrating that the former is a solution t the latter. To do this we substitute from (25) to (28) to get

$$\lambda(t) = \int_t^T A(s)[F_{ix}(s,x(s,u),u(s)) + \int_t^T F_{ix}(\tau,x(\tau,u),u(\tau)r(\tau,s)d\tau]ds + F_{ix}(t,x(t,x(t,u),u(t)) . \tag{29}$$

Application of (18) to the double integral yields

$$\lambda(t) = \int_t^T [A(s,t) + \int_t^s A(\tau,t)r(s,\tau)d\tau]F_{ix}(s,x(s,u),u(s))ds + F_{ix}(t,x(t,u),u(t)) . \tag{30}$$

Substitution for the bracketed term in (30) from (17) yields

$$\lambda(t) = F_{ix}(t,x(t,u),u(t)) + \int_t^T F_{ix}(s,x(s,u),u(s))r(s,t)ds \tag{31}$$

which is exactly (25), with s replaced by t, and τ by s.

5. DISCUSSION AND CONCLUDING COMMENTS

Consider a duopolistic industry in which the production cost of each firm depends on the experience it already gained. The standard procedures to model experience (see Spence (1981), and Stokey (1986)) is to define the accumulated output of each firm and to use it as a proxy for experience.[3] Under such an assumption, if we let $q_i(\tau)$ be the output of firm i at time τ, and define $Q(t) = Q(0) + \int_0^t q(\tau)d\tau$ and then assume that production cost is given by C(q, Q) then the experience effect simply implies that as Q rises, product cost declines. Clearly in this case the above integral equation can be restated such that $dQ/dt = q$ and if there is depreciation we can assume that $dQ/dt = q - \delta Q$. Note that the experience in this case can be described as some form of a capital acquired by the firm.

The above formulation implies that it is possible to gain experience in one day provided that the firm produces enough in such a day.

Let us now define the state variable experience E(t) as

$$E(t) = \int_0^t \alpha(q(s))e^{-f(s)(t-s)}ds$$

where $\alpha(q(s))$ is an increasing and concave function. The above integral equation is not trivial. We also assume that f(s) is an increasing function and thus the experience gained in the early stages depreciates faster than the experience gained in the later stages.

In discussing entry deterrence, the emphasis is on the asymmetry between the incumbent firm and the potential entrant. One such asymmetry is that the incumbent invested already in capital, capacity, technology or experience. Clearly from such an analysis it shows that such investments are important in evoking an entry barrier.

Discussing the interaction as integral games sheds some light on an important aspect of such interactions. When an incumbent invests in some capital and the entrant enters at a later period, the post-entry game is such that the players do not have the same type of capital. One has an old fast depreciating capital, while the entrant has a new modern one.

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NOTES

1. In the same manner one can define difference games when the time setting is discrete. For a discussion on differential and difference games see Basar and Olsder (1982).

2. We define U_i as time autonomous. This assumption can be generalized such that the admissible controls at time t are history dependent.

3. If there are externalities in learning, then the experience of each firm depends on the vector of accumulated output.