Optimal Control with Integral State Equations

M. I. KAMIEN and E. MULLER
Graduate School of Management, Northwestern University

Methods of optimal control theory have proved useful in studying the class of dynamic economic problems that can be posed as optimization of

$$\int_0^T F(t, x(t), u(t))dt \qquad \text{...(1)}$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0, \qquad \text{...(2)}$$

where t denotes time and $x(t)$, $\dot{x}(t)$, $u(t)$, the values of the state variable, its derivative, and the control variable respectively at time t, see [2] and [4]. Restriction of the state equation to a differential equation, as in (2), appears to deny the use of this technique for the analysis of other problems, such as the capital replacement problem investigated by Arrow [1],

$$\max_{I(t)} \int_0^T \alpha(t)[P(k(t), t) - I(t)]dt \qquad \text{...(3)}$$

s.t.

$$k(t) = \int_0^t M(t-s)I(s)ds. \qquad \text{...(4)}$$

In this problem $P(k(t), t)$ is the operating profit from stock of capital $k(t)$ at time t, $I(s)$ gross investment, $\alpha(t)$ the discount factor, $\rho(t) = -\dot{\alpha}(t)/\alpha(t)$ the instantaneous rate of interest, $m(u-s)$ the mortality density, i.e. the fraction of gross investment made at time s, that disappears about time u, and $M(t)$ is the mortality rate. From the last definitions it is evident that $\dot{M}(t) = -m(t)$ and that $m(u-s) \equiv 0$ if $u<s$. Expressions (3) and (4) constitute a control problem with investment $I(s)$ the control variable and capital stock $k(t)$, the state variable. The familiar state equation of capital theory,

$$\dot{k}(t) = I(t) - \delta K(t)$$

is a special case of (4) that obtains when $m(u-s) = \delta e^{-\delta(u-s)}$. See Mann [7] for further applications of the state equation (4) and some special forms of the mortality rate. It is interesting to note that the necessary conditions for the Nerlove-Arrow model, which are derived later in this paper, can be directly applied to Mann's paper with some minor modifications. The three models studied by Mann can be solved by substituting the specific mortality rate into the necessary conditions.

Arrow employed characteristic ingenuity to convert the problem posed in (3) and (4) into one suitable for analysis by calculus of variations. In this paper we will present a theorem similar to Bakke's [3] and Vinokurov's [11], which is suitable for dealing with such problems in a more straightforward, systematic fashion.

A second theorem indicating circumstances under which the necessary conditions are also sufficient follows. The capital replacement problem is then solved and an extension of the Nerlove-Arrow [9] result regarding the optimal ratio of advertising goodwill to sales is then presented.

Formally, the general problem is to maximize

$$\int_a^b F(t, x(t), u(t))dt \qquad \ldots(5)$$

subject to

$$x(t) = \int_a^t f(t, x(s), u(s), s)ds + x(a). \qquad \ldots(6)$$

Appending (6) to (5) with the usual Lagrange multiplier yields

$$\mathscr{L}(x, u, \lambda) = \int_a^b F(t, x(t), u(t))dt + \int_a^b \lambda(t)\left[\int_a^t f(t, x(s), u(s), s)ds + x(a) - x(t)\right]dt, \qquad \ldots(7)$$

changing the order of integration in (7) yields

$$\mathscr{L}(x, u, \lambda) = \int_a^b F(t, x(t), u(t))dt + \int_a^b \left[\int_t^b \lambda(\tau)f(\tau, x(t), u(t), t)d\tau + x(a)\lambda(t) - x(t)\lambda(t)\right]dt. \qquad \ldots(8)$$

Equation (8) can be written as

$$\mathscr{L}(x, u, \lambda) = \int_a^b H(t, x, u, \lambda)dt - \int_a^b \lambda(t)[x(t) - x(a)]dt, \qquad \ldots(9)$$

where the Hamiltonian $H(t, x, u, \lambda)$ is defined by

$$H(t, x, u, \lambda) = F(t, x(t), u(t)) + \int_t^b f(s, x(t), u(t), t)\lambda(s)ds. \qquad \ldots(10)$$

By a well-known result of the calculus of variations, the condition for $\mathscr{L}(x, u, \lambda)$ to be stationary is that

$$\partial H/\partial u = 0 = F_u(t, x(t), u(t)) + \int_t^b f_u(s, x(t), u(t), t)\lambda(s)ds \qquad \ldots(11a)$$

$$\partial H/\partial x = \lambda(t) = F_x(t, x(t), u(t)) + \int_t^b f_x(s, x(t), u(t), t)\lambda(s)ds. \qquad \ldots(11b)$$

This provides a rather heuristic proof for the following theorem:

Theorem 1. *If the requirements for the existence of a unique continuous solution to the integral equation* (6), *which include continuity of the function f together with a Lipschitz condition, are met; see* [8, *pp.* 24-30]. *In addition if the partial derivatives,* $\partial f/\partial x$, $\partial f/\partial u$, $\partial F/\partial x$, $\partial F/\partial u$ *exist and are continuous and* $f(t, x, u, s) \equiv 0$ *for* $t < s$ *then there exists a continuous multiplier function of time,* $\lambda(t)$ *and a Hamiltonian* $H(t, x, u, \lambda)$ *defined by* (10) *which satisfy* (11a) *and* (11b).

For a rigorous proof see [3] or [11].

Upon integrating the necessary conditions (11a) and (11b) two observations can be made:

(a) In case that $\partial f(t, x(s), u(s), s)/\partial t \equiv 0$ Theorem 1 reduces to the familiar Pontryagin principle.

(*b*) The multiplier $\lambda(t)$ is the time derivative of the multiplier of the Pontryagin principle with a state equation of the form (2).

The last observation readily gives rise to interpretation of the multiplier function, in this case, as the marginal shadow price of the state variable.

In order to derive the sufficient condition, let $u(t, x, \lambda)$ be the solution to max $H(t, x, u, \lambda)$; if $H(t, x, u(t, x, \lambda), \lambda)$ is a concave function of x, then so is the function

$$L_{t,\lambda}(x, u(t, x, \lambda)) = H(t, x, u(t, x, \lambda), \lambda) - \lambda(t)(x(t) - x(a)). \qquad \ldots(12)$$

If (11a) and (11b) are satisfied at x^*, u^*, and u^* is unique given x^*, so that $u^* = u(t, x^*, \lambda)$ then for all x and u,

$$L_{t,\lambda}(x, u) \leqq L_{t,\lambda}(x^*, u^*). \qquad \ldots(13)$$

Integrating (13) yields

$$\mathscr{L}(x, u, \lambda) \leqq \mathscr{L}(x^*, u^*, \lambda), \qquad \ldots(14)$$

when $\mathscr{L}(x, u, \lambda)$ is defined by (7). On feasible paths, the second integral in (7) vanishes and therefore

$$\int_a^b F(t, x(t), u(t))dt \leqq \int_a^b F(t, x^*(t), u^*(t))dt.$$

The following theorem summarizes this result.

Theorem 2. *Let $u(t, x, \lambda)$ be the solution to* $\max H(t, x, u, \lambda)$, *where H is defined by* (10). *If $H(t, x, u(t, x, \lambda), \lambda)$ is a concave function of the state variable x, if x^*, y^* satisfy* (11a) *and* (11b) *and $u^* = u(t, x^*, \lambda)$, then the necessary conditions are also sufficient for a maximum of* (5) *subject to* (6).

See [5] for the corresponding theorem in the case of a differential state equation.

In order to solve Arrow's capital replacement problem, we form the Hamiltonian of (3) and (4):

$$H = \alpha(t)[P(k(t), t) - I(t)] + I(t) \int_t^T [M(s-t)]\lambda(s)ds \qquad \ldots(15)$$

and the necessary conditions are

$$\partial H/\partial I = -\alpha(t) + \int_t^T [M(s-t)]\lambda(s)ds = 0 \qquad \ldots(16)$$

$$\partial H/\partial k = \lambda(t) = \alpha(t)P_k[k(t), t]. \qquad \ldots(17)$$

Differentiation of (16) yields a linear integral equation of the convolution type in $\lambda(t)$

$$\lambda(t) = -\dot{\alpha}(t) + \int_0^T m(s-t)\lambda(s)ds \qquad \ldots(18)$$

since $m(s-t) \equiv 0$ for $s<t$. The solution of which is

$$\lambda(t) = -\dot{\alpha}(t) - \int_0^T r(s)\dot{\alpha}(s-t)ds, \qquad \ldots(19)$$

where $r(s)$ denotes the resolvent kernel, see [8, pp. 13-21], that can be shown to be identical to the replacement density $r(s)$ which was defined by Arrow to be

$$r(s) = \textstyle\sum_1^\infty m^{(n)}(s),$$

where $m^{(n)}(s)$, called the nth convolution of $m(s)$, is defined recursively by

$$m^{(1)}(s) = m(s),\ m^{(n+1)}(s) = \int_0^s m^{(n)}(t)m(s-t)dt.$$

Substitution into (17) and division of both sides by $\alpha(t)$ yields

$$P_k[k(t), t] = -\dot{\alpha}(t)/\alpha(t) - \int_0^T \frac{r(s)}{\alpha(t)}\dot{\alpha}(s-t)ds. \qquad \ldots(20)$$

Recalling the definition of $\rho(t)$ gives rise to a finite horizon version of Arrow's myopic rule for optimal capital investment

$$\begin{aligned} P_k[k(t), t] &= \rho(t) + \int_0^T [r(s)\rho(s-t)\alpha(s-t)/\alpha(t)]ds \\ &= \rho(t) + \bar{r}(t), \end{aligned} \qquad \ldots(21)$$

where $\bar{r}$, implicitly defined by (21), may be interpreted as the average number of replacements from t forward. While the above analysis only demonstrates that results obtained by more traditional methods can be duplicated by application of Theorem 1, it also provides a vehicle for generalizations that could not be as conveniently treated by those techniques. For example, a mortality density that is dependent on the contemporaneous level of the capital stock $m(t-s, k(s))$ could be accommodated by the Maximum Principle, as could other generalizations recently suggested by Malcomson [6] and Nickell [10].

Before concluding, we indicate an extension of the Nerlove-Arrow [9] result regarding the optimal ratio of advertising goodwill to sales. In their paper revenue $P(k(t), t)$ is $pq(p, k)-c(q)$, where p denotes product price, demand $q(p, k)$ is a function of both price and advertising goodwill k, and $c(q)$ denotes production cost. We assume that advertising goodwill is accumulated through current expenditure, $I(t)$, in accordance with (4) rather than with the special case of Nerlove-Arrow in which the mortality rate follows an exponential law. The maximization problem has the same form as (3) and (4) except for the addition of product price as a control variable. Necessary conditions for this problem are

$$\lambda(t) = -\dot{\alpha}(t)+\int_0^T m(s-t)\lambda(s)ds \qquad \text{...(22)}$$

$$\alpha(t)[q+(p-c')\partial q/\partial p] = 0 \qquad \text{...(23)}$$

$$\lambda(t) = \alpha(t)(p-c')\partial q/\partial k. \qquad \text{...(24)}$$

If $\alpha(t) \neq 0$, then (23) implies $(p-c') = -q(\partial q/\partial p)$ which upon substitution into (24), multiplication of both sides by k/pq, and rearrangement of terms yields:

$$\beta\alpha(t)/\eta\lambda(t) = k/pq, \qquad \text{...(25)}$$

where $\beta = (k\partial q/\partial k)/q$, $\eta = -(p\partial q/\partial p)/q$ are the elasticities of demand with respect to goodwill and price, respectively. Recollection of (17), (20) and (21) and substitution for $\lambda(t)/\alpha(t)$ in (25) yields the desired result:

$$k/pq = \beta/\eta[\rho(t)+\bar{r}(t)]. \qquad \text{...(26)}$$

Expression (26) specializes to the marginal condition obtained by Nerlove-Arrow under the supposition of exponential decay of goodwill at rate δ, for then $\bar{r} = \delta$. According to (26) the ratio of goodwill to sales revenue along an optimal policy is directly related to the elasticity of demand with respect to goodwill and inversely to the price elasticity of demand, the instantaneous rate of interest, and the average anticipated decay rate from the present forward.

Summary. A simplified version of a theorem of Bakke for optimal control with an integral state equation has been presented. Usual concavity conditions have been shown to render the necessary conditions for an optimum sufficient as well. Finally, application of this Maximum Principle to some capital replacement problems has demonstrated how they might be treated in a uniform fashion.

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