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Characterization of Constant Policies in Optimal Control[1]

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Abstract. The conditions under which an optimal control problem commonly employed in economics gives rise to a constant optimal control are characterized. The conditions are stated in terms of the properties of the functional form of the integrand of the objective function and of the state equation. These conditions can be checked prior to computation of the optimal control and thereby can simplify its calculation.

Key Words. Optimal control problems, current value Hamiltonian, constant optimal controls, analytic functions, calculus of variations, differential games.

1. Introduction

The focus of most dynamic economic models is analysis of their steady state behavior. The steady state is characterized by constancy of the optimal control as well as of the state variable and the costate variable. The steady state solution is in general not optimal but is approached by the optimal solution in the limit as time approaches infinity. There may be circumstances, however, when the optimal control is constant through time from the outset, rather than just in the limit. Our objective is to characterize the circumstances under which the optimal control is constant through time in terms of the properties of the integrand of the objective function and of the state equation. With these characterizations, it is possible to determine if the optimal control will be constant without actually going through all the computations required to find the optimal control. Moreover, knowledge that the optimal control will be constant simplifies determination of its actual value.

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2. Characterization of Constant Policies in Optimal Control

It is well known that extremals of a calculus of variations problem of the form

$$\max \int_{t_0}^{t_1} f(x'(t))\, dt, \tag{1a}$$

$$\text{s.t.} \quad x(t_0)=x_0, \qquad x(t_1)=x_1, \tag{1b}$$

are linear functions of time; see Kamien and Schwartz (Ref. 1). This means that, for the equivalent optimal control problem,

$$\max \int_{t_0}^{t_1} f(u(t))\, dt, \tag{2a}$$

$$\text{s.t.} \quad x'(t)=u(t), \qquad x(t_0)=x_0, \qquad x(t_1)=x_1, \tag{2b}$$

the optimal control $u^*(t)$ is constant through time, that is, $du^*/dt=0$. We know, however, that constancy of the optimal control through time is not confined to this case. For example, consider the limit pricing problem:

$$\max \int_0^\infty \exp(-rt)[R_1(p(t))(1-F(t))+R_2(t)F(t)]\, dt, \tag{3a}$$

$$\text{s.t.} \quad F'(t)=h(p(t))(1-F(t)), \qquad F(0)=0; \tag{3b}$$

here, r is the discount rate; $p(t)$, the control variable, is the price chosen by a monopolistic firm at time t; $R_1(p(t))$ is its profit realized prior to rival entry; $R_2(t)$ is its profit realized after rival entry; $F(t)$, the state variable, is the probability of rival entry on or before time t; and $h(p(t))$ is the instantaneous conditional probability of rival entry at time t, given no prior entry; for this problem, the optimal price is constant through time; see Kamien and Schwartz, pp. 206–208 (Ref. 1).

It is our objective to determine the general circumstances under which the control is constant through time for the class of optimal control problems

$$\max \int_0^\infty \exp(-rt) f(x(t), u(t))\, dt, \tag{4a}$$

$$\text{s.t.} \quad x'(t)=g(x(t), u(t)), \qquad x(0)=x_0, \tag{4b}$$

where $u(t)$ is the control variable and $x(t)$ is the state variable, $x'(t)=dx(t)/dt$. This formulation is commonly employed in intertemporal models in economics and management science. We assume that this problem has a solution and that the necessary conditions are also sufficient. We assume further that $f(x(t), u(t))$ and $g(x(t), u(t))$ are analytic in $u(t)$ and twice differentiable in $x(t)$, that $g(x(t), u(t))$ does not vanish for all values of $u(t)$ and $x(t)$, and that $f(x(t), u(t))$ and $g(x(t), u(t))$ are linearly independent, i.e.,

$$f(x(t), u(t)) \neq cg(x(t), u(t)), \qquad \text{for some constant } c.$$

In the absence of the last assumption, the optimal control would coincide with the static maximum of $f(x(t), u(t))$ and would therefore be constant, but for a trivial situation. We seek the functional forms of $f(x(t), u(t))$ and $g(x(t), u(t))$ that cause the optimal control for (4) to be constant.

The current value Hamiltonian for (4) is

$$H(x(t), u(t), m(t)) = f(x(t), u(t)) + m(t)g(x(t), u(t)), \tag{5}$$

where $m(t)$ is the current value multiplier. The necessary conditions for a maximum of (4) are that

$$H_u(x(t), u(t), m(t)) = f_u(x(t), u(t)) + m(t)g_u(x(t), u(t)) = 0, \tag{6}$$

where the subscript indicates the partial derivative with respect to that variable, and

$$\begin{aligned} m'(t) &= rm(t) - H_x(x(t), u(t), m(t)) \\ &= rm(t) - f_x(x(t), u(t)) - m(t)g_x(x(t), u(t)), \end{aligned} \tag{7}$$

where the prime refers to differentiation with respect to time. Differentiation of expression (6) with respect to time yields

$$H'_u = H_{uu}u' + H_{ux}x' + H_{um}m' = 0. \tag{8}$$

It is evident from expression (8) that, since $H_{uu} < 0$ is assumed, $u' = 0$, if and only if

$$H_{ux}x' + H_{um}m' = 0. \tag{9}$$

Now, from expression (6), it follows that $H_{um} = g_u \neq 0$; for, if $g_u = 0$, for all u, the control would not influence the behavior of the state variable, and problem (4) would degenerate into a static optimization problem. Execution of the required differentiation and substitution from expressions (4), (6), (7) into (9) yields

$$rmg_u + (gf_{ux} - g_u f_x) + m(gg_{ux} - g_u g_x) = 0, \tag{10}$$

or equivalently

$$(m' + f_x + mg_x)g_u + (gf_{ux} - g_u f_x) + m(gg_{ux} - g_u g_x) = 0. \tag{11}$$

Expressions (10) and (11) give rise to three cases leading to satisfaction of expression (9).

Case 1. $r = 0$ and $gf_{ux} - g_u f_x + m(gg_{ux} - g_u g_x) = 0$.

Case 2. $m = 0$ and $gf_{ux} - g_u f_x = 0$.

Case 3. $m' = 0$ and $g_u f_{ux} - f_u g_{ux} = 0$.

Here, use has been made of the fact that $m = -f_u/g_u$. We shall analyze each of these cases in turn.

3. Results for Case 1

Theorem 3.1. Suppose that $f(x(t), u(t))$ and $g(x(t), u(t))$ are nonlinear functions of their arguments. Then, the following three conditions are equivalent.

(a) The optimal control $u^*(t)$ is constant through time.
(b) $u^*(t)$ maximizes $f/|g|$.
(c) either (i) f and g are of the form

$$f(x(t), u(t)) = f_1(x(t))f_2(u(t)),$$

$$g(x(t), u(t)) = g_1(x(t))g_2(u(t)),$$

and the equation $f_2' g_2 = f_2 g_2'$ has a solution; or (ii) for some nonseparable function $h(x(t), u(t))$,

$$f/g = h_1(x(t))h_2(u(t))(u(t) - u^*)^2 h(x(t), u(x)) + k(x(t)).$$

Proof. Define

$$y(x(t)) = g(x(t), u^*)/g_u(x(t), u^*),$$

$$a(x(t)) = f(x(t), u^*), \qquad b(x(t)) = f_u(x(t), u^*).$$

Then, upon recollection that $m = -f_u/g_u$, Case 1 amounts to satisfaction of the differential equation

$$y' + (b'/b)y - a'/b = 0, \tag{12}$$

where the prime here indicates differentiation with respect to x. The solution of this differential equation involves two possibilities:

$$(A)\ y' \neq 0 \quad \text{or} \quad (B)\ y' = 0.$$

(A) For the case $y' \neq 0$, the solution to (12) consists of a general solution to the homogeneous equation and a particular solution to the nonhomogeneous equation. The particular solution is $y = a/b$, while the general solution is $y=-k/b$ for some constant k. Thus, the general solution to (12) is

$$y=(a-k)/b, \tag{13}$$

which, upon recollection of the definitions of a and b, yields

$$g/g_u=(f-k)/f_u. \tag{14}$$

The assumption that a solution to (4) exists implies that $k=0$. Moreover, cross multiplication by f_u in (14) and recollection that $m=-f_u/g_u$ discloses that k is just the Hamiltonian correspoding to (4) and therefore that it is identically zero in this case.

With $k=0$, (14) can be rewritten as

$$g(x, u^*)f_u(x, u^*)-f(x, u^*)g_u(x, u^*)=0, \tag{15}$$

which, upon division by $g^2(x, u^*)$, yields

$$\partial(f/g)/\partial u=(gf_u-fg_u)/g^2=0. \tag{16}$$

Thus, u^* is a stationary point for $f(x, u)/g(x, u)$. Moreover,

$$\partial^2(f/g)/\partial u^2|_{u^*}=(gf_{uu}-g_{uu}f)/g^2=H_{uu}/g<0. \tag{17}$$

Hence, if $g>0$, u^* maximizes f/g; and, if $g<0$, u^* minimizes f/g. In other words, u^* maximizes $f/|g|$.

(B) For the case $y'=0$, we have from (12) and the definitions of a and b that

$$g/g_u=f_x/f_{ux}; \tag{18}$$

but (18), together with (10), implies also that

$$g_u/g=g_{ux}/g_x, \qquad \text{at } u^*. \tag{19}$$

Now, (19) only involves the function g. A small change in the value of f will cause a continuous change in u^* according to (18). Thus, (19) must hold in a small neighborhood of u^* as well. Integrating (19) with respect to u yields

$$\log(g)=\log(g_x)+\log(k(x)), \tag{20}$$

or

$$g/g_x=k(x), \tag{21}$$

where $k(x)$ is some function of x. Thus, the ratio g/g_x is only a function of x. Integrating (21) with respect to x yields

$$g=g_1(x)g_2(u), \tag{22}$$

for some functions g_1 and g_2. Now, substitution from (22) into the left-hand side of (18) discloses that the ratio f_x/f_{ux} is only a function of u. Integration of (18) with respect to u gives

$$\log(f_x) = \log(g) + k(x), \tag{23}$$

or

$$f_x = g \exp[k(x)], \tag{24}$$

for some function $k(x)$. Integration of (24) with respect to x yields

$$f = f_1(x) f_2(u) + f_3(u). \tag{25}$$

But the existence of a solution to (4) implies $f_3(u) \equiv 0$. Thus,finally we have

$$f(x, u) = f_1(x) f_2(u). \tag{26}$$

For the forms of g and f given by (22) and (26) respectively, (18) becomes

$$f_2' g_2 - g_2' f_2 = 0, \tag{27}$$

that is, the first-order condition for maximization of f/g. It is straightforward to check that f/g is indeed maximized at u^*.

To show the equivalence of parts (a) and (c) of Theorem 3.1, we note that, in (B), f and g are multiplicatively separable and that this is equivalent to (A), where they are not. In (A), however, $\partial(f/g)/\partial u$ vanishes at $u = u^*$. Since f and g are assumed to be analytic in u, their ratio f/g is as well. Thus, f/g can be expanded into a Taylor series with respect to u. As the first derivative of f/g vanishes at $u = u^*$, the result is a function of the form appearing in (B), where h_1 and h_2 are the separable parts (if any) of the nonlinear part of the Taylor series expansion. □

4. Results for Case 2

Theorem 4.1. If $m = 0$ and $f_{ux} g - g_u f_x = 0$, then

$$f(x, u) = h(x, u)(u - u^*)^n, \qquad \text{for } n > 1,$$

and $g(x, u) = k(x) f_x(x, u)$, for some functions h and k, and $u' = 0$.

Proof. From the necessary condition (6), it follows that, if $m = 0$, then $f_u = 0$, at $u = u^*$. Now,

$$f_u = h_u(x, u)(u - u^*)^n + nh(x, u)(u - u^*)^{n-1} = 0, \qquad \text{at } u = u^*, \text{ for } n > 1.$$

Also, division of the differential equation in Case 2 by g^2 yields

$$(gf_{ux}-g_uf_x)/g^2=d(f_x/g)/du=0, \tag{28}$$

which, upon integration with respect to u, yields

$$f_x/g=k(x), \tag{29}$$

for some function $k(x)$, or

$$g=k(x)f_x. \tag{30}$$

We observe also that

$$f_x=h_x(u-u^*)^n,$$

so that, by (7),

$$m'=-f_x=0, \qquad \text{at } u=u^*,$$

and, by (4),

$$x'=g=k(x)f_x=0, \qquad \text{at } u=u^*, \text{ for } n>1. \qquad \square$$

If $n=1$, f and g are just functions of x, and we have the uninteresting case in which the control does not enter into the problem.

5. Results for Case 3

Theorem 5.1. If $f_u/g_u=h(u)$, then $u'=0$ if and only if $m'=0$.

Proof. Under the hypothesis of the theorem, the differential equation

$$g_uf_{ux}-g_{ux}f_u=0 \tag{31}$$

must be satisfied (see statement of Case 3). Division of (36) by g_u^2 yields

$$(g_uf_{ux}-g_{ux}f_u)/g^2=d(f_u/g_u)/dx=0, \tag{32}$$

which implies that

$$f_u/g_u=h(u), \tag{33}$$

for some function $h(u)$. But, by (6), $h(u^*)=-m$. Thus, if u^* is constant, then m is constant; and, if m is constant through time, then u^* must be constant through time. $\square$

It is easy to check that the limit pricing problem (3) satisfies the conditions of Theorem 5.1. The current value Hamiltonian for this problem is

$$H=R_1(p)(1-F)+R_2F+mh(p)(1-F), \tag{34}$$

and so the necessary conditions for a maximum are

$$\partial H/\partial P = R_1'(p)(1-F) + mh'(p)(1-F) = 0, \tag{35}$$

$$\partial^2 H/\partial P^2 = (R_1'(p) + mh''(p)](1-F) \leq 0, \tag{36}$$

$$m' = rm - \partial H/\partial F = R_1(p) - R_2 + m(r + h(p)]. \tag{37}$$

It follows from (35) that

$$m = -R_1'(p)/h'(p); \tag{38}$$

therefore, if p is constant through time, then so is m. The value of m can be derived from (37) and the transversality condition

$$\lim_{t\to\infty} m(t)F(t) = 0. \tag{39}$$

Specifically, (37) can be rewritten as

$$m' - m(r + h(p)) = R_1(p) - R_2. \tag{40}$$

Multiplication of both sides by the integrating factor $\exp[-(r+h(p))t]$, integration, and appeal to the transversality condition (39) for evaluation of the constant of integration yields

$$m(t) = -[R_1(p) - R_2]/(r + h(p)]. \tag{41}$$

Moreover, it is easy to check that the differential equation (31) is satisfied for this problem.

6. Extension

The same type of analysis can be applied to differential games as summarized by the following theorem.

Theorem 6.1. If $f(x, u, v) = f_0(x)f_1(u)f_2(v)$ and $g(x, u, v) = g_0(x)g_1(u)g_2(v)$, and if the equations $f_1'g_1 = g_1'f_1$ and $f_2'g_2 = f_2g_2'$ have solutions, then the closed-loop Nash equilibrium solution of the symmetric noncooperative differential game,

$$\max_{u(t)} \int_0^\infty \exp(-rt)f(x(t), u(t), v(t))\, dt, \tag{42a}$$

$$\text{s.t.} \quad x' = g(x(t), u(t), v(t)), \qquad x(0) = x_0, \tag{42b}$$

$$\max_{v(t)} \int_0^\infty \exp(-rt)f(x(t), v(t)\, dt, \tag{42c}$$

is constant through time, where $u(t)$ is the first player's control and $v(t)$ the second player's.

Proof. The Hamiltonian for the first player's problem is

$$H = f(x, u, v) + mg(x, u, v), \tag{43}$$

and the necessary conditions for a Nash equilibrium are (see Starr and Ho, Ref. 2)

$$H_u = 0, \tag{44}$$

$$m' = -H_x - H_v\, \partial v^*/\partial x. \tag{45}$$

By symmetry, the identical problem is faced by the second player.

Differentiation of (44) with respect to time yields that $u' = 0$ if and only if

$$H_{uv}v' + H_{ux}x' + H_{um}m' = 0. \tag{46}$$

Substitution from (42) for x', from (45) for m', from (44) for m, and evaluation of the coefficients in (46), with f and g as specified, yields the stipulated result. It is straightforward to show that, in our case, $H_v = 0$; and so, the closed-loop and open-loop equilibria for this problem coincide. □

See Reinganum (Ref. 3) for a different class of problems that possesses the same property.

7. Conclusions

We have characterized the circumstances under which the optimal control to a commonly employed dynamic optimization model in economics and management science is constant through time. We have done this is in terms of simple conditions on the integrand $f(x(t), u(t))$ and the state equation $g(x(t), u(t))$.

These conditions can easily be checked prior to calculation of the optimal control. Thus, if they are satisfied, then we know immediately that the optimal control is constant. These conditions also suggest the circumstances when constant growth of the money supply, assuming that it is the control variable in an intertemporal macroeconomic optimization model, would be optimal. A recent analysis of an optimal growth model in which the rate of growth of the money supply is the control variable has been conducted by Drabicki and Takayama (Ref. 4). They find that the optimal rate of growth of the money supply is not constant through time. Analysis of their model discloses that their formulation fails to satisfy the theorems presented here.

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