

# Social welfare functions when preferences are convex, strictly monotonic, and continuous

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## Abstract

The paper shows that if the class of admissible preference orderings is restricted in a manner appropriate for economic and political models, then Arrow's impossibility theorem for social welfare functions continues to be valid. Specifically if the space of alternatives is  $R_+^n$ ,  $n \geq 3$ , where each dimension represents a different public good and if each person's preferences are restricted to be convex, continuous, and strictly monotonic, then no social welfare function exists that satisfies unanimity, independence of irrelevant alternatives, and nondictatorship.

Arrow (1963) proved that for a set of at least three alternatives no non-dictatorial social welfare function (SWF) exists satisfying unanimity (U) and independence of irrelevant alternatives (IIA), provided admissible preferences are not *a priori* restricted in some manner. If, however, the variety of preference orderings that are admissible is restricted sufficiently, then non-dictatorial SWFs do exist that satisfy U and IIA. Single-peakedness, which Black (1948) discovered and Arrow (pp. 75-80) discussed, is the best known of these restrictions that is sufficient to make majority rule into a non-dictatorial transitive SWF satisfying U and IIA. Papers of Inada (1969) and of Sen and Pattanaik (1969) generalized single-peakedness and determined necessary and sufficient restrictions on the set of admissible preferences for majority rule to be a transitive SWF satisfying Arrow's conditions. Kramer (1973) used these results to show that majority rule is a valid Arrow type SWF only if the set of admissible preferences is restricted to a class that is much smaller than is justifiable by economic or political theory.

These results describe the properties only of majority rule. The power of Arrow's theorem is that it rules out construction of any nondictatorial SWF satisfying U and IIA, not just social welfare functions based on majority rule. Our purpose in this paper is to show that the negative conclusions derived for the special case of majority rule generalize into true impossibility results.<sup>1</sup>

The specific situation that we wish to explore is as follows. Society is a

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set  $I = \{1, \dots, m\}$  of  $m$  individuals. Each individual  $i$  has a complete, transitive, and reflexive preference ordering  $\lesssim_i$  over the set of possible public good consumption bundles  $\alpha$ . Let  $N = \{1, \dots, n\}$  represent the  $n$  different public goods within the society. Each element  $x = (x^1, \dots, x^n)$  of  $\alpha$  is a public goods bundle, i.e.  $x$  is a  $n$ -vector where  $x^i$  is the quantity of the  $i$ th public good. Generally we define  $\alpha \equiv R_+^n$ , the nonnegative orthant of  $n$ -dimensional Euclidean space. An individual's preferences  $\lesssim_i$  depends jointly on the intrinsic structure of  $\alpha$  and on his tastes. For example, suppose individuals are not satiated with public goods. If each component  $x^j$  of  $x$  is greater than the corresponding component  $y^j$  of  $y$ , then  $x$  dominates  $y$  and every individual is certain to strictly prefer  $x$  to  $y$ . In other words, every individual's preference relation agrees with the dominance relation that is intrinsic to  $\alpha$ . If, however, neither  $x$  dominates  $y$  nor  $y$  dominates  $x$ , then whether an individual prefers  $x$  to  $y$  or  $y$  to  $x$  is a matter of personal taste that may vary from individual to individual. Consequently the structure of  $\alpha$  limits the orderings  $\lesssim_i$  that are admissible as individual  $i$ 's preferences. Let  $\theta$  represent this set of admissible preference orderings.  $\theta$  is unsubscripted because, for this public goods case, the intrinsic structure of  $\alpha$  is invariant from individual to individual.

The purpose of a SWF is to be a rule for aggregating the profile  $(\lesssim_1, \dots, \lesssim_m)$  of individuals' preferences over  $\alpha$  into a complete, transitive, and reflexive societal preference ordering  $\lesssim$  over  $\alpha$ . Since tastes can not be predicted *a priori*, the SWF, in order to be applicable in all situations, must be defined for all possible profiles  $(\lesssim_1, \dots, \lesssim_m)$  that are consistent with the intrinsic structure of  $\alpha$ . In other words, the SWF must be defined for every profile  $(\lesssim_1, \dots, \lesssim_m)$  for which  $\lesssim_1 \in \theta, \lesssim_2 \in \theta, \dots, \lesssim_m \in \theta$ , i.e. its domain must be  $\theta^m$ , the  $m$ -fold cartesian product of  $\theta$ .

The question therefore is this. Given a set  $\alpha$  and the set  $\theta$  of admissible preference orderings that  $\alpha$ 's structure implies, can a nondictatorial SWF be constructed whose domain is  $\theta^m$  and which satisfies Arrow's conditions U and IIA.<sup>2</sup> As stated above, the set  $\alpha$  that we consider is  $R_+^n$  where each axis represents a distinct public good. We assume that individuals' preferences for these public goods have the characteristic that economists normally ascribe to preferences for economic goods, be they private or public goods. These characteristics are three:

1. the technical assumption that an individual's preferences can be represented by a continuous utility function;
2. that individuals are insatiable, i.e. the utility function is monotonic, and
3. that individuals' indifference surfaces are convex from below.

These three assumptions mean that every individual's indifference map is of the conventional type. We prove that if  $\theta$  is restricted to preference orderings satisfying (1) through (3), then no nondictatorial SWF exists that has

$\theta^m$  as its domain and satisfies U and IIA. Moreover the method of proof makes clear that even much more rigorous restrictions on  $\theta$  than (1)-(3) still leads to impossibility results. Therefore our conclusion is that Arrow's assumption of unrestricted preferences played an innocent role in his impossibility result.

**1. Formulation and examples**

Let  $\Sigma$  be the set of all complete reflexive, and transitive preference relations defined over the alternative set  $\alpha$ . The set of admissible preferences  $\theta$  is a subset of  $\Sigma$ . Individual  $i$ 's preferences are represented by  $\lesssim_i \in \theta$ . The symbolism  $x \succ_i y$  denotes strict preference of alternative  $x \in \alpha$  over alternative  $y \in \alpha$ . Similarly  $x \sim_i y$  denotes indifference and  $x \succsim_i y$  denotes preference or indifference. Two preference relations  $\lesssim$  and  $\lesssim'$  are said to agree on a subset  $B$  of  $\alpha$  if, for every pair  $x, y \in B$ ,  $x \lesssim y$  if and only if  $x \lesssim' y$ . We denote agreement on  $B$  by  $\lesssim|_B = \lesssim'|_B$ . Two profiles,  $\lesssim_I = (\lesssim_1, \dots, \lesssim_m)$  and  $\lesssim'_I = (\lesssim'_1, \dots, \lesssim'_m)$ , agree on  $B \subset \alpha$  if, for all  $i \in I$ ,  $\lesssim_i|_B = \lesssim'_i|_B$ .

A SWF on  $\theta$  is a function  $f: \theta^m \rightarrow \Sigma$ . Notice that the range of a SWF is not restricted to  $\theta$ . An Arrow SWF is a SWF that satisfies the conditions of unanimity and independence of irrelevant alternatives.

*Unanimity (U).* Let  $\lesssim_I \in \theta^n$  be any admissible profile and let  $f(\lesssim_I) = \lesssim$ . The SWF  $f$  satisfies U if and only if, for any pair of alternatives  $x, y \in \alpha$ ,  $x \prec_i y$  for all  $i \in I$  implies  $x < y$

*Independence of Irrelevant Alternatives (IIA).* Let  $\lesssim_I \in \theta^m$  and  $\lesssim'_I \in \theta^m$  be any two admissible profiles. Let  $f(\lesssim_I) = \lesssim$  and  $f(\lesssim'_I) = \lesssim'$ . The SWF  $f$  satisfies IIA if and only if, for any subset  $B \subset \alpha$ ,  $\lesssim_I|_B = \lesssim'_I|_B$  implies  $\lesssim|_B = \lesssim'|_B$

A SWF has a dictator on the set  $B \subset \alpha$  if and only if an individual  $i \in I$  exists such that, for every profile  $\lesssim_I = (\lesssim_1, \dots, \lesssim_m) \in \theta^m$  and every pair of alternatives  $x, y \in B$ ,  $y \prec_i x$  implies  $y < x$  where  $\lesssim = f(\lesssim_I)$ . A family  $\theta$  is called dictatorship enforcing if every Arrow SWF on  $\theta^n$  has a dictator on the full alternative set  $\alpha$ .

*Example A (Arrow's Theorem).* If  $|\alpha| \geq 3$  and if  $\theta = \Sigma$ , then  $\theta$  is dictatorship enforcing.<sup>3</sup>

*Example B.* The family  $\theta^2_+$  of linear monotonic preference relations, defined on  $R^2_+$ , the nonnegative quadrant of two dimensional Euclidean space, is not dictatorship enforcing. Formally, a preference relation  $\lesssim$  is contained in  $\theta^2_+$  if a scalar  $a > 0$  exists such that, for any pair,  $x, y \in R^2_+$ ,  $x \lesssim y$  if and only if  $ax^1 + x^2 \geq ay^1 + y^2$ .

Given that  $\theta_2^+$  is the family of admissible preference relations, let the preferences  $(\lesssim_1, \dots, \lesssim_m)$  of the  $m$  individuals within the society be described by the vector  $(a_1, \dots, a_m)$  where  $a_i$  is the parameter that describes the linear preferences of person  $i$ . Finally let  $a = A_M(a_1, \dots, a_m)$  be the median value of the vector  $(a_1, \dots, a_m)$ . A valid nondictatorial Arrow SWF defined on the family  $\theta_2^+$  is this:  $\lesssim = f_M(\lesssim_1, \dots, \lesssim_m)$  where  $\lesssim$  is that linear preference relation whose parameter is  $A_M(a_1, \dots, a_m)$ . For an extensive discussion of this case for majority rule, see Nitzan (1976).

*Example C.*<sup>4</sup> Let  $\alpha$  contain six elements:  $\{x, x', x'', y, y', y''\}$  and let  $X = \{x, x', x''\}$  and  $Y = \{y, y', y''\}$ . Define:

$$\theta = \{\lesssim \in \Sigma \mid w \in X \text{ and } z \in Y \text{ implies } w > z\} \tag{1}$$

i.e.  $\lesssim$  is admissible if and only if every element of the triple  $X$  is ranked above every element of the complementary triple  $Y$ . This class  $\theta$  is not dictatorship enforcing because Arrow's theorem applies separately to the two triples but not to both jointly. Specifically a nondictatorial Arrow SWF may be constructed as follows: (1) make individual one dictator over  $X$ , (2) make individual two dictator over  $Y$ , and (3) make the social ordering  $\lesssim$  rank all elements of  $X$  above all elements of  $Y$ . All remaining individuals are dummies.

*Example D.* Alter example C by reducing the size of  $\theta$  :

$$\theta = \{\lesssim \in \Sigma \mid [y > y' > y''] \text{ and } [w \in X \text{ and } z \in Y \text{ implies } w > z]\} \tag{2}$$

i.e.  $\lesssim$  is admissible only if every element of the triple  $X$  is ranked above every element of the triple  $Y$  and the elements of the triple  $Y$  are ranked in the descending order  $y, y'$  and  $y''$ . This class is dictatorship enforcing because Arrow's theorem applies to the triple  $X$  and condition U applies to the triple  $Y$  since  $\theta$  fixes preferences over the triple  $Y$ .

Examples C and D are important because they provide a counterexample to the common misconception that making  $\theta$  smaller always makes construction of a nondictatorial Arrow SWF easier.

## 2. A useful theorem

In this section we state a simple theorem that is useful in determining whether any particular  $\theta \subset \Sigma$  is a dictatorship enforcing class of preference

relations. Throughout  $\theta$  represents a fixed, nonempty subset of  $\Sigma$ . A pair of distinct alternatives  $x, y \in \alpha$  is called trivial (relative to  $\theta$ ) if all the relations in  $\theta$  agree on the set  $\{x, y\}$ . A set of three distinct alternatives  $\{x, y, z\}$  is called a free triple if (1) each alternative is distinct (i.e.  $x \neq y \neq z \neq x$ ) and (2) for every  $\succsim \in \Sigma$ , there exists  $\succsim' \in \theta$  such that:

$$\succsim| \{x, y, z\} = \succsim'| \{x, y, z\} \tag{3}$$

In other words,  $\{x, y, z\}$  is a free triple if  $\theta$  admits all possible orderings of the three alternatives. Two nontrivial pairs  $B = \{x, y\}$  and  $C = \{c, z\}$  are called strongly connected if  $|B \cup C| = 3$  and  $B \cup C$  is a free triple. Thus  $B$  and  $C$  are strongly connected if they share an element in common and together form a free triple. Two pairs  $B$  and  $C$  are called connected if a finite sequence of pairs:

$$B = B_1, B_2, \dots, B_{n-1}, B_n = C \tag{4}$$

exist such that  $B_i$  and  $B_{i+1}$  are strongly connected for each  $i = 1, 2, \dots, n-1$ . Finally a class  $\theta$  is called saturating if (a) the set  $\alpha$  contains at least two nontrivial pairs and (b) every nontrivial pair  $B \subset \alpha$  is connected to every other nontrivial pair  $C \subset \alpha$ .

*Theorem 1.* Every saturating class  $\theta$  is dictatorship enforcing.

Examples C and D illustrate this theorem’s usefulness. In example C  $\theta$  is not saturating because the two triples  $X$  and  $Y$  are both free, but are not connected. Therefore Theorem 1 is inapplicable and, as shown before, a nondictatorial, Arrow SWF can be constructed. In example D  $\theta$  is saturating because only  $X$  is a free triple. All pairs involving the triple  $Y$  are trivial. Therefore Theorem 1 implies that  $\theta$  is dictatorship enforcing.

*Proof of Theorem 1*

The first of four steps is to show that if a nontrivial pair  $B$  is strongly connected to another nontrivial pair  $C$ , then an individual  $j \in I$  exists who is dictator on  $D = B \cup C$ . Since, by hypothesis,  $B$  and  $C$  are strongly connected,  $D$  is a free triple. Arrow’s theorem may be applied to this triple: an individual  $j \in I$  exists who is a dictator on  $D$ . The second step is to note that if an individual  $j \in I$  is dictator on a pair  $B_i$  and a second pair  $B_{i+1}$  exists to which  $B_i$  is strongly connected, then  $j$  is also dictator on  $B_{i+1}$ . The third step is to note that if two pairs  $B$  and  $C$  are connected, then step two implies that an individual  $j \in I$  exists who is dictator on both. The last step is to note two facts. First, because  $\theta$  is saturating, at least two nontrivial pairs exist and each is connected with every other, nontrivial pair. Consequently, an individual  $j$  exists who is dictator over them. Second,

if a pair  $B = \{x, y\}$  is trivial, then individual  $j$ , along with every other individual  $i \in I$ , is dictator on  $B$ . Hence individual  $j$  is dictator on all pairs, trivial and nontrivial. Q.E.D.

### 3. Convex, continuous, strictly monotonic preferences

In this section we use Theorem 1 to show that the class  $\theta_n^*$  of all convex, continuous, and strictly monotonic preference relations defined on  $R_+^n$ , the nonnegative orthant of  $n$ -dimensional, Euclidean space, is dictatorship enforcing. A preference relation is convex if, for every alternative  $x \in R_+^n$ , the set  $\{y \in R_+^n \mid x \succsim y\}$  is convex. A family  $\theta_n \subset \Sigma$  is convex if every  $\succsim \in \theta_n$  is convex. A preference relation  $\succsim \in \Sigma$  is continuous if it can be represented by a continuous utility function on  $R_+^n$ . A preference relation  $\succsim \in \Sigma$  is strictly monotonic if, for any pair of distinct alternatives,  $x, y \in R_+^n$ ,  $x < y$  implies  $x < y$ .<sup>5</sup>

*Theorem 2.* The class  $\theta_n^*$  of convex, strictly monotonic, continuous preference relations on  $R_+^n$  is dictatorship enforcing for all  $n \geq 1$ .

#### *Proof of Theorem 2*

If  $n = 1$ , then  $\theta_1^*$  consists of one element and every individual is a dictator. If  $n \geq 3$ , then the proof, without any loss of generality, may be constructed using only linear preference relations. If  $n = 2$ , then the proof is somewhat more difficult; it employs the same strategy but requires the use of a non-linear class of convex, strictly monotonic, continuous preference relations. Therefore we first spell out the proof for the  $n \geq 3$  case and then sketch the proof for the  $n = 2$  case. The proof for the  $n \geq 3$  case is in three steps. It consists of showing that  $\theta_n^*$  is saturating and therefore dictatorship enforcing.

*Step 1.* A preference relation  $\succsim \in \Sigma$  is linear if and only if a vector  $p = (p_1, \dots, p_n) \in R_+^n$  exists such that, for all pairs  $(x, y) \in R_+^n$ ,  $x \succsim y$  if and only if  $\langle p, x \rangle \leq \langle p, y \rangle$  where  $\langle p, x \rangle = \sum p_i x^i$ , the inner product of  $p$  and  $x$ . Three observations follow directly from this definition. First, if a linear preference relation is parameterized by the vector  $p \in R^n$ , the indifference surface containing a specific point  $x' \in R_+^n$  is the plane  $\{x \in R_+^n \mid \langle p, x \rangle = \langle p, x' \rangle\}$ . Second, every linear preference relation is convex. Third, a linear preference relation  $\succsim$  with parameter vector  $p \in R^n$  is strictly monotonic if and only if  $p \gg 0$ .

*Step 2.* A pair  $(x, y) \in R_+^{2n}$ ,  $x \neq y$ , is nontrivial if and only if neither  $x > y$  nor  $y > x$ . If  $x > y$ , then strict monotonicity implies that  $y < x$  for all  $\succsim \in \theta_n^*$ . Identical reasoning applies to the  $x < y$  case. Therefore if  $x > y$  or  $y > x$ , the pair  $(x, y)$  is trivial. If neither  $x < y$  nor  $y > x$ , then a pair of components  $(i, j) \in N \times N$  must exist such that  $x^i > y^i$  and  $x^j < y^j$ . Linear preference relations  $\succsim' \in \theta_n^*$  and  $\succsim'' \in \theta_n^*$  exist such that  $x <' y$  and

$y <'' x$ . These two relations are constructed by showing that a linear  $\lesssim \in \theta^*$  exists such that  $x \sim y$  and then perturbing  $\lesssim$  slightly to obtain  $\lesssim'$  and  $\lesssim''$ .

To show that a linear  $\lesssim \in \theta_n^*$  exists such that  $x \sim y$  it is necessary to find a vector  $p = (p_1, \dots, p_n) \in R^n, p \geq 0$ , such that:

$$\langle p, x \rangle = \langle p, y \rangle \tag{5}$$

Because neither  $x > y$  nor  $y > x$ , a pair of indices  $(i, j) \in N \times N$  exists such that  $x^i > y^i$  and  $x^j < y^j$ . Impose the restriction, without loss of generality, that  $\sum p_i = 1$ . Equation (5) may therefore be solved for  $p_i$ :

$$p_i = \frac{[1 - \sum_{\substack{k \neq i \\ k \neq j}} p_k](y^j - x^j) + \sum_{\substack{k \neq i \\ k \neq j}} p_k(y^k - x^k)}{[(x^i - y^i) + (y^j - x^j)]} \tag{6}$$

The denominator is positive and the numerator can be made positive by packing each component  $p_k$  ( $k = 1, 2, \dots, n; k \neq i, k \neq j$ ) such that it is positive and sufficiently close to zero. Therefore  $p_i$  can be made positive and, consequently a  $p \in R^n$  exists such that  $p \geq 0$  and (5) is satisfied.

Given that a relation  $\lesssim \in \theta_n^*$  exists such that  $x \sim y$ , a relation  $\lesssim' \in \theta_n^*$  may easily be constructed such that either  $x <' y$  or  $y <' x$ . For example, in order to construct  $\lesssim'$  such that  $x <' y$ , pick a point  $x^* \gg x$  that preserves the inequalities  $x^{*i} > y^i$  and  $x^{*j} < y^j$ . Construct, as above, a linear  $\lesssim' \in \theta_n^*$  such that  $x^* \sim' y$ . Strict monotonicity and transitivity then implies that  $x <' y$ . Therefore the claim that  $(x, y) \in R^{2n}$  is a nontrivial pair if and only if neither  $x > y$  nor  $y > x$  is true.

*Step 3.* Any nontrivial pair  $(x, y) \in R_+^{2n}$  is connected to a reference pair  $(e_1, e_2) \in R_+^{2n}$ . Let these reference points be the unit vectors  $e_1 = (e_1^1, \dots, e_1^n) \in R^n$  and  $e_2 = (e_2^1, \dots, e_2^n) \in R^n$  where  $e_i = (e_i^1, \dots, e_i^n) \in R^n$  has the property that  $e_i^j = 1$  if  $i = j$  and  $e_i^j = 0$  if  $i \neq j$ .

*Observation 1.* If  $(x, y)$  is a nontrivial pair, then a linear  $\lesssim \in \theta_n^*$  exists such that  $x \sim y$ . We proved this observation immediately above in the proof's second step.

*Observation 2.* If a linear  $\lesssim \in \theta_n^*$  exists such that  $w \sim x \sim y$  for a triple  $(w, x, y) \in R_+^{3n}$  of non-collinear points, then  $(w, x, y)$  is a free triple. Given that  $w \sim x \sim y$  for  $\lesssim \in \theta_n^*$ , an ordering  $\lesssim' \in \theta_n^*$  such that  $w <' x <' y$  may be constructed as follows. Pick points  $w^* \in R_+^n$  and  $y^* \in R_+^n$  such that  $w^* \gg w, y \gg y^*$ , and the distances  $\|w^* - w\|$  and  $\|y - y^*\|$  are small. If  $w^*$  and  $y^*$  are chosen close enough to  $w$  and  $y$  respectively, then continuity guarantees that a  $\lesssim'' \in \theta_n^*$  exists such that  $w^* \sim'' x \sim'' y^*$ . Consequently, by

transitivity and monotonicity,  $w <'' x <'' y$ .

*Observation 3.* If a triple  $(w, x, y) \in R^{3n}$  is composed of points that lie on a distinct axis, then a linear  $\lesssim \in \theta_n^*$  exists such that  $w \sim x \sim y$ . Without loss of generality, let  $w = \zeta_w e_1$ ,  $x = \zeta_x e_2$ , and  $y = \zeta_y e_3$  where  $\zeta_w, \zeta_x$ , and  $\zeta_y$  are strictly positive scalars. If  $\lesssim' \in \theta_n^*$  is a linear ordering parameterized by the vector  $p = (p_1, \dots, p_n)$  where  $p_1 = \zeta_w^{-1}$ ,  $p_2 = \zeta_x^{-1}$ ,  $p_3 = \zeta_y^{-1}$ , and  $p_k = 1$  ( $k = 4, 5, \dots, n$ ), then it satisfies the requirement  $w \sim' x \sim' y$ .

Given these three observations we can show that any nontrivial pair  $(x, y)$  is connected to the reference pair  $(e_1, e_2)$ . Observation 1 states that a linear  $\lesssim \in \theta_n^*$  exists such that  $x \sim y$ . Let  $p$  be the vector that parameterizes  $\lesssim$ . Pick an index  $i \in N$  and a point  $z_1 = \zeta_1 e_i$  on axis  $i$  such that:

- a.  $\langle p, x \rangle = \langle p, z_1 \rangle = \langle p, y \rangle$  and
- b.  $x, y$ , and  $z_1$  are not colinear

i.e.  $x \sim y \sim z_1$ . Such a pair  $i \in N$  and  $z_1 \in R_+^n$  exists because  $p \gg 0$  and  $n \geq 3$ . In fact,  $\zeta_1 = \langle p, x \rangle \div p_i$ . Observation 2 implies that  $(x, y, z_1)$  is a free triple.

The construction that led to the choice of  $z_1$  implies that  $(y, z_1)$  is a nontrivial pair. Therefore, in exactly the same manner that we picked the index  $i$  and the point  $z_1$ , we may pick a second, distinct index  $j \in N$ , a point  $z_2 = \zeta_2 e_j$ , and a vector  $p > 0$  such that:

- a.  $\langle p, y \rangle = \langle p, z_1 \rangle = \langle p, z_2 \rangle$  and
- b.  $y, z_1$ , and  $z_2$  are not colinear

Therefore  $(y, z_1, z_2)$  is a free triple.

The points  $z_1$  and  $z_2$  are nontrivial. Therefore, as before, pick an index  $k \in N$ , a point  $z_3 = \zeta_3 e_k$ , and a vector  $p > 0$  such that:

- a.  $\langle p, z_1 \rangle = \langle p, z_2 \rangle = \langle p, z_3 \rangle$ ,
- b.  $z_1, z_2$ , and  $z_3$  are not colinear,
- c.  $k = 1$  if  $i \neq 1$  and  $j \neq 1$ ,
- d.  $k = 2$  if  $\{i \neq 1 \text{ or } j \neq 1\}$  and  $\{i \neq 2 \text{ and } j \neq 2\}$ , and
- e.  $k = 3$  otherwise

The triple  $(z_1, z_2, z_3)$  is free.

By construction an  $l \in \{1, 2, 3\}$  exists such that  $z_l = \zeta_l e_1$ . Without loss of generality suppose that  $z_3 = \zeta_3 e_1$ . By construction  $z_1 \neq \zeta_1 e_2$  or  $z_2 \neq \zeta_2 e_2$ . Suppose, again without loss of generality, that  $z_2 \neq \zeta_2 e_2$ . Let  $z_4 = e_2$ , the second reference point. Pick a vector  $p \gg 0$  such that:



$$\langle p, z_2 \rangle = \langle p, z_3 \rangle = \langle p, z_4 \rangle \tag{7}$$

Observation 3 guarantees that this construction is possible. Since  $z_2, z_3,$  and  $z_4$  all lie on different axes they cannot be colinear. Therefore  $(z_2, z_3, z_4)$  is a free triple. Let  $z_5 = e_1$ , the first reference point. Observation 3 states that a vector  $p \geq 0$  exists such that  $\langle z_2, p \rangle = \langle z_4, p \rangle = \langle z_5, p \rangle$ . Therefore  $(z_2, z_4, z_5)$  is a free triple.

The product of this procedure is the following collection of free triples:  $(x, y, z_1), (y, z_1, z_2), (z_1, z_2, z_3), (z_2, z_3, z_4),$  and  $(z_2, z_4, z_5)$ . From this collection a sequence of pairs may be extracted:  $B_1 = (x, y), B_2 = (y, z_1), B_3 = (z_1, z_2), B_4 = (z_2, z_3), B_5 = (z_2, z_4),$  and  $B_6 = (z_4, z_5) = (e_1, e_2)$ . Inspection shows that the pairs  $B_i$  and  $B_{i+1}$  are strongly connected for  $i = 1, 2, \dots, 5$ . Thus the terminal pairs  $(x, y)$  and  $(e_1, e_2)$  are connected. Therefore every nontrivial pair is connected to the reference pair and the family  $\theta_n^*$  is saturating. Consequently, by Theorem 1,  $\theta_n^*$  is dictatorship enforcing for  $n \geq 3$ .

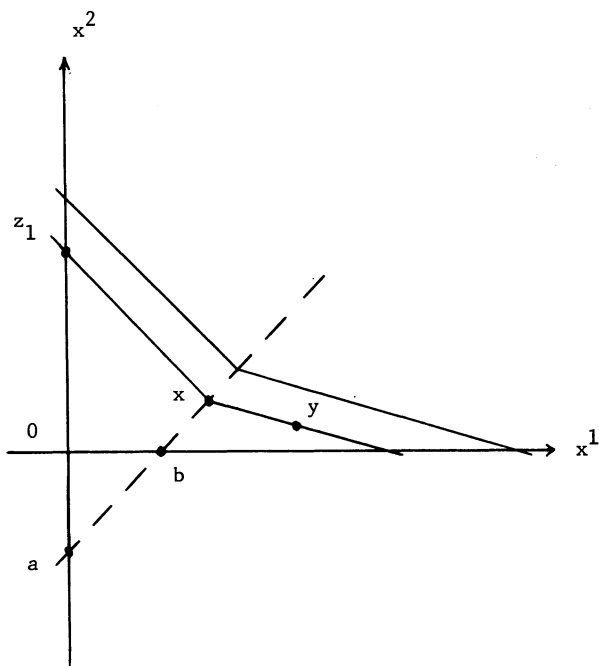
The case of  $n = 2$  may be proved using the same program of showing that every nontrivial pair  $(x, y) \in R_+^4$  is connected to the reference pair  $(e_1, e_2)$ . The difference is that when  $n = 2$  linear preference relations cannot be used to show that a point  $z_1 \in R_+^2$  exists such that  $(x, y, z_1)$  is a free triple. The family  $\theta_2$  of piecewise linear preference relations, however, can be used to show that  $(x, y)$  is contained within a free triple and therefore can be used to prove the theorem for  $n = 2$ . The preference relation  $\lesssim \in \Sigma_2$  is an element of  $\theta_2$  if and only if a vector  $q = (q_1, q_2, q_3, q_4) \in R^4$  exists such that, for all pairs  $(x, y) \in R_+^4, x \lesssim y$  if and only if:

$$\begin{aligned} q_1 x^1 + q_2(x^2 + q_4) + q_3 \text{Min}[x^1, x^2 + q_4] &\leq \\ q_1 y^1 + q_2(y^2 + q_4) + q_3 \text{Min}[y^1, y^2 + q_4] &\tag{8} \end{aligned}$$

If  $q_1 > 0, q_2 > 0,$  and  $q_3 > 0,$  then  $\lesssim \in \theta_2$  is both convex and strictly monotonic. Figure 1 shows the type of family of indifference curves that an element of  $\theta_2$  generates. In the figure the elements of the triple  $(x, y, z_1)$  are indifferent with each other. If we perturb the elements of  $(q_1, q_2, q_3, q_4),$  then the indifference curves can be shifted sufficiently to achieve any desired ordering of  $(x, y, z_1);$  therefore  $(x, y, z_1)$  is a free triple. Given this technique for constructing free triples, the remainder of the proof for the  $n = 2$  case exactly parallels the proof for the  $n \geq 3$  case. Q.E.D.

#### 4. Linear preferences

Example B described a nondictatorial Arrow SWF for the case where  $\alpha = R_+^2$  and  $\theta = \theta_2^+,$  the class of linear strictly monotonic preference relations. Inspection of Theorem 2's proof, which for  $n \geq 3$  depended only on linear



*Note.* The lengths of the line segments  $Oa$  and  $Ob$  are equal to the value of the parameter  $q_4$  (in drawing the diagram we have assumed a positive value for  $q_4$ ). The region below the dotted diagonal contains all points  $x = (x^1, x^2)$  such that  $x^1 > x^2 + q_4$ . The segment  $zx$  of the indifference curve has slope  $-(q_1 + q_3)/q_2$  and the segment  $xy$  of the indifference curve has slope  $-q_1/(q_2 + q_3)$ .

*Figure 1.*

preference relations, shows that example B is not generalizable to dimensions higher than two. Therefore the following corollary is true.

*Corollary.* The class  $\theta_n^+$  of linear preference relations on  $R_+^n$  is dictatorship enforcing for all  $n \geq 3$ .

The interest of this result is that restricting admissible preferences to be linear is a strong assumption that is generally unjustifiable. Nevertheless, when  $\alpha$  is at least three dimensional, even that is not sufficiently strong to allow construction of a nondictatorial Arrow SWF.

## Notes

1. Subsequent to the original writing of this paper we learned that Maskin (1976) was also working on this question using a different approach. His paper considers the case of purely public goods. His proof relies on necessary and sufficient conditions for  $\theta$  to be dictatorship enforcing. This contrasts with our proof which relies on Theorem 1's relatively simple sufficient condition for  $\theta$  to be dictatorship enforcing. The result he obtains for private goods exactly parallels the result we obtain for public goods.
2. Papers of Kalai and Muller (1977) and of Maskin (1976) have separately developed necessary and sufficient conditions that characterize those classes of admissible preferences for which a nondictatorial social welfare function satisfying U and IIA exists. The condition developed here is implied by their conditions.
3. The notation  $|\alpha|$  denotes the number of elements in the set  $\alpha$ .
4. For a general discussion of this type of example, see Fishburn (1976).
5. The notation  $x > y$  means that each component of the vector  $x$  is at least as great as the corresponding component of vector  $y$  and at least one component of  $x$  is strictly greater than the corresponding component of  $y$ . The notation  $x \gg y$  means that every component is strictly greater than every component of  $y$ .

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